CONSTRUCTING MATRICES WITH GIVEN GENERALIZED CHARACTERISTIC POLYNOMIAL

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ABSTRACT

Generalized characteristic polynomials, a standard tool in 2-D control theory, are treated from a symbolic point of view. The main cases of the multidimensional realization problem can be handled, even with parameters rather than explicite numbers, up to dimension at least 5 with a recent implementation of Janet's algorithm. For the bivariate case a general existence result is presented. The problem to parametrize all realizations can be split up into a sequence of smaller problems, each of which can be tackled with the software available. We concentrate on the generic cases. Various coefficients in the generalized characteristic polynomial are interpreted as (generalized) characteristic polynomials of submatrices, and their expansion in multivariate Lagrange interpolation bases turns out to be relevant.

1. INTRODUCTION

The present paper describes an approach to find matrices with given generalized characteristic polynomial as described in [3], [4] by means of symbolic methods. The problem comes from the realization problem of a multi-dimensional control system described by the Roesser model. The computational methods employed go back to M. Janet and were recently taken up again in the context of Groebner basis theory, cf. [5] or [6] for a theoretical background, [2] for an algorithmic background, and [1] for a package, where the algorithms are realized.

In Section 2 a general existence theorem for the bivariate case is proved (under a weak technical assumption to simplify the proof) and the use of the Janet package is demonstrated to find all realizations in sufficiently small examples. In the example given, the polynomial is given by parameters. If one takes explicit numbers one can go much further. Section 3, the extended bivariate case, which is actually relevant in the original problem, is dealt with. It is demonstrated that matrices up to degree 5 can easily be handled.

2. THE BIVARIATE CASE

Let $m, n \in \mathbb{N}$ be natural numbers and $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}, D \in \mathbb{C}^{m\times m}$ be matrices. The unit matrix of degree n ist denoted by I_n and the block diagonal matrix of degree $n+m$ with diagonal blocks A, D by $A \oplus D$. Finally the generalized characteristic polynomial of

$$H := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathbb{C}^{(n+m)\times(n+m)}$$

is defined as

$$\chi_H^{(n,m)}(s,t) := \det(sI_n \oplus tI_m - H) \in \mathbb{C}[s,t]$$

Clearly, $\chi_H^{(n,m)}(s,s) = \chi_H(s)$ is the characteristic polynomial of H. The degree in t of $\chi_H^{(n,m)}(s,t)$ is m, its degree in s is n and the coefficient of $s^n t^m$ is equal to 1. Clearly, any such polynomial $\psi(s,t) \in \mathbb{C}[s,t]$ can uniquely be written as

$$\psi(s,t) \quad = \quad \alpha(s)\delta(t) + \rho(s,t) \qquad (1)$$

with $\delta(t) \in \mathbb{C}[t]$ the coefficient of s^n in $\psi(s,t) \in \mathbb{C}[t][s]$ and $\alpha(s) \in \mathbb{C}[s]$ the coefficient of t^m in $\psi(s,t) \in \mathbb{C}[s][t]$. Clearly $\delta(t)$ is monic of degree m, $\alpha(s)$ is monic of degree n and $\rho(s,t) \in \mathbb{C}[s,t]$ is of degree at most $n-1$ in s and at most $m-1$ in t. In case $\psi(s,t) = \chi_H^{(n,m)}(s,t)$ one has $\delta(t) = \chi_D(t) \in \mathbb{C}[t]$, the characteristic polynomial of D and $\alpha(s) = \chi_A(s) \in \mathbb{C}[s]$. We call ψ generic, if both $\delta(t)$ and $\alpha(s)$ have no multiple roots.

Proposition 2.1 *Let $\psi \in \mathbb{C}[s,t], n,m \in \mathbb{N}$ as above and assume that ψ is generic, then there exists a matrix $H \in \mathbb{C}^{(n+m)\times(n+m)}$ such that $\psi(s,t) = \chi_H^{(n,m)}(s,t)$*

Proof: Let $\alpha(s) = \prod_{i=1}^{n}(s-a_i)$ and $\delta(t) = \prod_{i=1}^{m}(t-d_j)$. We define $A := \mathrm{Diag}(a_1,\ldots,a_n) \in \mathbb{C}^{n\times n}$ and $D := \mathrm{Diag}(d_1,\ldots,d_m) \in \mathbb{C}^{m\times m}$ as diagonal matrices having $\alpha(s)$ resp. $\delta(t)$ as their characteristic polynomials. Moreover let $C \in \mathbb{C}^{m\times n}$ have all its entries equal to 1 and denote with I_n the identity matrix of dimension n. We have to solve the equation

$$\det\left(\begin{array}{c|c} sI_n - A & -B \\ \hline -C & tI_m - D \end{array}\right) = \psi(s,t)$$

by comparing the coefficients of the monomials in s and t as equations for the $b_{i,j}$. Not only will these equations be linear, there will be nm such equations and they will be linearly independent, leaving us with exactly one solution for $B \in \mathbb{C}^{n\times m}$. To see this, view the determinant as polynomial in the $b_{i,j}$. For instance the coefficient for $b_{1,1}$ is up to sign the determinant of the matrix obtained from the above matrix by omitting the first row and the $(n+1)$st column. Subtracting the first column from the second to n-th column and then subtracting the $(n+1)$st from the $(n+2)$nd to $(n+m)$th row, shows that this determinant is $\pm\prod_{i=2}^{n}(s-a_i)\prod_{j=2}^{m}(t-d_j)$. By symmetry, the same analysis applies to the other $b_{i,j}$. Let

$$\lambda_i^{(a)}(s) := \prod_{k=1,k\neq i}^{n}(s-a_k) \quad (2)$$

$$\lambda_j^{(d)}(t) := \prod_{l=1,l\neq j}^{m}(t-d_j) \quad (3)$$

$$\Lambda_{i,j}^{(a),(d)} := \lambda_i^{(a)}(s)\lambda_j^{(d)}(t) \quad (4)$$

with $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Since the $\lambda_i^{(a)}(s)$ form the Lagrange interpolation basis for the polynomials in $\mathbb{C}[s]$ of degree less than n and correspondingly the $\lambda_j^{(d)}(t)$ the Lagrange interpolation basis for the polynomials in $\mathbb{C}[t]$ of degree less than m , the $\Lambda_{i,j}^{(a),(d)}$ form a basis of the space of polynomials of degree smaller than n in s and less than m in t in $\mathbb{C}[s,t]$. Keeping track of the signs and taking the term independent of the $b_{i,j}$ into account, which one gets by computing the above determinant for $b_{i,j} = 0$ for all i,j yields the following expression for the determinant:

$$\prod_{k=1}^{n}(s-a_k)\prod_{l=1}^{m}(t-d_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}\Lambda_{i,j}^{(a),(d)}b_{i,j}.$$

Because the $\Lambda_{i,j}$ form a basis, $\rho(s,t) = \psi(s,t) - \alpha(s)\delta(t)$ can be uniquely expressed in the $\Lambda_{i,j}$ and the coefficients give us the negative values of the $b_{i,j}$ and we have their uniqueness together with the desired existence. q.e.d.

Since it is a standard linear algebra task, to expand univariate polynomials in the Lagrange interpolation basis. The twovariate case is reduced to the univariate one in a straightforward matter. Here is an example.

Example 2.2 $\psi(s,t) := s^3t^3 + 3\,s^3t^2 + 2\,s^3t - 3\,s^2t^3 - 9\,s^2t^2 - 6\,s^2t + 2\,t^3s + 6\,t^2s + 6\,ts + s^2 + 3\,s + t^2 + t + 3$ *yields* $\alpha(s) = s(s-1)(s-2)$ *and* $\beta(t) = t(t+1)(t+2)$ *and hence* $a = (0,1,2)$ *and* $d = (0,-1,-2)$. *Hence* $\psi(s,t) - \alpha(s)\beta(t) = s^2 + (3+2\,t)\,s + t^2 + t + 3$. *Expressing the s^i in the interpolation basis* ($\lambda_1^{(a)} := (s-1)(s-2), \lambda_2^{(a)} := s(s-2), \lambda_3^{(a)} := s(s-1)$) *and their coefficients in the interpolation basis* ($\lambda_1^{(d)} := (t+1)(t+2), \lambda_2^{(d)} := t(t+2), \lambda_3^{(d)} := t(t+1)$) *yields*

$$\begin{aligned}
&\psi(s,t) - \alpha(s)\beta(t) \\
&= (-\lambda_2^{(a)} + 2\lambda_3^{(a)})(\frac{1}{2}\lambda_1^{(d)} - \lambda_2^{(d)} + \frac{1}{2}\lambda_3^{(d)}) \\
&+(-\lambda_2^{(a)} + \lambda_3^{(a)})(\frac{3}{2}\lambda_1^{(d)} - \lambda_2^{(d)} - \frac{1}{2}\lambda_3^{(d)}) \\
&+(\frac{1}{2}\lambda_1^{(a)} - \lambda_2^{(a)} + \frac{1}{2}\lambda_3^{(a)})(\frac{3}{2}\lambda_1^{(d)} - 3\lambda_2^{(d)} + \frac{5}{2}\lambda_3^{(d)}) \\
&= -\sum b_{i,j}\Lambda_{i,j}^{(a),(d)}
\end{aligned}$$

and hence

$$\begin{aligned}
-(b_{i,j}) &= \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \\
&+ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \end{pmatrix} \\
&+ \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -3 & \frac{5}{2} \end{pmatrix} \\
&= -\begin{pmatrix} -3/4 & 3/2 & -5/4 \\ 7/2 & -5 & 5/2 \\ -\frac{13}{4} & 9/2 & -7/4 \end{pmatrix}
\end{aligned}$$

Proposition 2.1 is an existence result, which does not really give an idea of how many matrix solutions to a given polynomial exist. Here is an attempt so parametrize at least algorithmically the variety of solutions.

Let $H \in \mathbb{C}^{(n+m)\times(n+m)}, \psi \in \mathbb{C}[s,t]$, $\alpha(s) = \prod_{i=1}^{n}(s-a_i)$, $\delta(t) = \prod_{i=1}^{m}(t-d_i)$. Call H with $\chi_H^{(n,m)}(s,t) = \psi(s,t)$ seminormalized, if $A = \mathrm{Diag}(a_1,\ldots,a_n)$ and $D = \mathrm{Diag}(d_1,\ldots,d_m)$. Among the seminormalized matrices H we first concentrate on those none of whose entries of $C = (c_{i,j})$ are equal to zero. From first principles we have the following.

Remark 2.3 *Let $\psi(s,t)$ be generic.*
1.) The group $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$ acts on

$$V(\psi) \;:=\; \{H = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathbb{C}^{(n+m)\times(n+m)} \;|\chi_H^{(n,m)}(s,t) = \psi(s,t)\}$$

by conjugation as the subgroup $T(n,m,\mathbb{C})$ of diagonal matrices does on

$$V_D(\psi) = \{\left(\begin{array}{c|c} \mathrm{Diag}(a_1,\ldots,a_n) & B \\ \hline C & \mathrm{Diag}(d_1,\ldots,d_m) \end{array}\right) \in \mathbb{C}^{(n+m)\times(n+m)} |\chi_H^{(n,m)}(s,t) = \psi(s,t)\}$$

2) Any $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$-orbit in $V(\psi)$ is represented in $V_D(\psi)$, i. e. the orbit spaces $V(\psi)/\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$ and $V_D(\psi)/T(n,m,\mathbb{C})$ can be identified.
3) The stabilizer of any $H \in V_D(\psi)$ with all $c_{i,j} \neq 0$ is $\mathbb{C}^* I_{n+m}$, even in $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$.
4) Any $H \in V_D(\psi)$ with all $c_{i,j} \neq 0$ has a unique matrix in its $T(n,m,\mathbb{C})$-orbit with $c_{1,i} = 1$ and $c_{j,1} = 1$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Such a representative will be called normalized.

Note, in the cases $n = 1$ or $m = 1$ all the normalized representatives can be determined by help of Proposition 2.1. To find all normalized representatives in case $n > 1$ and $m > 1$ one has to solve algebraic rather than linear equations.

Example 2.4 *Let $n = m = 2$. We try to find all normalized representatives of $V(\psi)$ with*

$$\begin{aligned} \psi(s,t) &:= \alpha(s)\delta(t) + \rho(s,t) \\ &= (s-a_1)(s-a_2)(t-d_1)(t-d_2) \\ &\quad + k_{0,0} + k_{1,0}s + k_{0,1}t + k_{1,1}st \end{aligned}$$

with $a_1 \neq a_2$ and $d_1 \neq d_2$. Then $A := \mathrm{Diag}(a_1,a_2)$, $D := \mathrm{Diag}(d_1,d_2)$. We use the abbreviation $\omega := (a_1-a_2)(d_1-d_2)$. First of all there is the easy case with $C := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. With Proposition 2.1 we get

$$B := \frac{1}{\omega}\begin{pmatrix} -\rho(a_1,d_1) & \rho(a_1,d_2) \\ \rho(a_2,d_1) & -\rho(a_2,d_2) \end{pmatrix}$$

The next case is $C := \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix}$ with $x \neq 0, 1$. In this case one gets with the Janet package [1]

$$B = \begin{pmatrix} xb_{2,2} + \frac{\rho(a_2,d_2)-\rho(a_1,d_1)}{\omega} & -xb_{2,2} + \frac{k_{1,0}+k_{1,1}d_2}{d_1-d_2} \\ -xb_{2,2} + \frac{k_{1,0}+k_{1,1}a_2}{a_1-a_2} & b_{2,2} \end{pmatrix}$$

where $b_{2,2}$ satisfies

$$\begin{aligned} 0 \;=\; & xb_{2,2}^2 \\ & + \left(\frac{\rho(a_2,d_2)-\rho(a_1,d_1)}{\omega} + \frac{xk_{1,1}}{x-1} + \frac{x\omega}{(x-1)^2}\right) b_{2,2} \\ & + \frac{(k_{1,0}+k_{1,1}d_2)(k_{0,1}+k_{1,1}a_2)}{(x-1)\omega} + \frac{\rho(a_2,d_2)}{(x-1)^2} \end{aligned}$$

For the case $C := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ one gets two situations. If $\rho(a_1,d_1) - \rho(a_2,d_2) \neq 0$ then one has the same solution as above with $x = 0$. In case $\rho(a_1,d_1) - \rho(a_2,d_2) = 0$ then

$$B = \begin{pmatrix} 0 & \frac{k_{1,0}+d_2k_{1,1}}{d_1-d_2} \\ -\frac{k_{1,0}+k_{1,1}d_1}{d_1-d_2} & b_{2,2} \end{pmatrix}$$

with $b_{2,2}$ arbitrary. If C has two and more zeroes, one only gets solutions for special $\psi(s,t)$ or more precisely $\rho(s,t)$.

3. THE EXTENDED BIVARIATE CASE

In the actual application, one has a slightly more complicated generalized characteristic polynomial: Given $G \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$ then the generalized $(n,m,1)$ -characteristic polynomial is defined as

$$\chi_G^{(n,m,1)}(s,t) := \det(sI_n \oplus tI_m \oplus z - G) \in \mathbb{C}[s,t,z]$$

The criterion for a polynomial $\psi(s,t,z) \in \mathbb{C}[s,t,z]$ of degree n in s, degree m in t and degree 1 in z with leading term $s^n t^m z$ is as follows. First denote the coefficient of $t^m z$ in $\psi(s,t,z)$ by $\alpha(s) \in \mathbb{C}[s]$, of $s^n z$ by $\delta(t) \in \mathbb{C}[t]$, and finally the coefficient of $s^n t^m$ by $z - f \in \mathbb{C}[z]$. Similarly as above call $\psi(s,t,z)$ generic, if neither $\alpha(s)$ nor $\delta(t)$ have multiple roots. Similarly to 1 one has

$$\psi(s,t,z) = \alpha(s)\delta(t)(z-f) + \rho(s,t,z) \tag{5}$$

where each monomial $s^i t^j z^k$ occurring in $\rho(s,t,z) \in \mathbb{C}[s,t,z]$ satisfies at least two of the conditions $i < n, j < m, k < 1$. If $\rho(s,t,z)$ only depends on two of the three variables, one easily adjusts the methods of Proposition 2.1 and obtains the existence of an $G \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$ with $\psi(s,t,z) = \chi_G^{(n,m,1)}(s,t,z)$ in the generic case. To discuss the general case of existence of $G \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$ let

$$G := \left(\begin{array}{c|c|c} A & B & V \\ \hline C & D & W \\ \hline v & w & f \end{array}\right) \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$$

with $A = \mathrm{Diag}(a_1,\dots,a_n)$, B, C, $D = \mathrm{Diag}(d_1,\dots,d_m)$ as earlier and $V \in \mathbb{C}^{n\times 1}$, $W \in \mathbb{C}^{m\times 1}$, $v \in \mathbb{C}^{1\times n}$, $w \in \mathbb{C}^{1\times m}$.

Remark 3.1 *The subspace of $\mathbb{C}[s,t,z]$ spanned by the monomials, which might occur in $\rho(s,t,z)$ is of dimension $2nm+n+m$. This differs from the number of free parameters in $G \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$ by $n+m$, i. e. the number of entries in B, V, C, W, v, w, i. e. in the description of*

$$\begin{aligned} V_D(\psi) \;&:=\; \{G \in \mathbb{C}^{(n+m+1)\times(n+m+1)} \\ &\quad |\chi_G^{(n,m,1)}(s,t,z) = \psi(s,t,z)\}. \end{aligned}$$

is equal to $2(nm+n+m)$.

Of these indeterminates some can usually be chosen resp. computed right at the beginning, namely v, w, V, W in such a way that one can hope to have (generically) only finitely many solutions for the remaining indeterminates.

Proposition 3.2 *Let $\psi(s,t,z) = \prod_{i=1}^n (s-a_i)\prod_{i=1}^m (t-d_i)(z-f) + \rho(s,t,z) \in \mathbb{C}[s,t,z]$ with ρ of total degree at most $n+m-1$, degree at most n in s, degree at most m in t, and 1 in z. Let $G \in \mathbb{C}^{(n+m+1)\times(n+m+1)}$ as above with $(n,m,1)$-characteristic polynomial $\chi_G^{(n,m,1)}(s,t,z) = \psi(s,t,z)$.*
1) Let $\psi_t(s,z)$ be the coefficient of t^m in ψ. It can be uniquely written as

$$\psi_t(s,z) = \prod_{i=1}^n (s-a_i)(z-f) - \sum_{i=1}^n p_i \lambda_i^{(a)}(s)$$

with $p_i \in \mathbb{C}$. Then $V_i v_i = p_i$ for all i. For each $p_i \neq 0$, one can assume after suitable conjugation that $v_i := 1$ and obtains $V_i = p_i$.
2) Let $\psi_s(t,z)$ be the coefficient of s^n in ψ. It can be uniquely written as

$$\psi_s(t,z) = \prod_{i=1}^m (t-d_i)(z-f) - \sum_{i=1}^n q_i \lambda_i^{(d)}(t)$$

for some $q_i \in \mathbb{C}$. One has $W_i w_i = q_i$ for all $i = 1,\dots,m$. In case $q_i \neq 0$, one may again set $w_i := 1$ and gets $W_i = q_i$.
3) Let $\psi_z(s,t)$ be the coefficient of z in ψ. It can be uniquely written as

$$\begin{aligned} \psi_z(s,t) \;&=\; \prod_{i=1}^n (s-a_i)\prod_{j=1}^m (t-d_j) \\ &\quad - \sum_{i=1}^n \sum_{j=1}^m r_{i,j}\lambda_i^{(a)}(s)\lambda_j^{(d)}(t) \end{aligned}$$

with unique $r_{i,j} \in \mathbb{C}$ and is the (n,m)-characteristic polynomial of the submatrix $H \in \mathbb{C}^{(n+m)\times(n+m)}$ of G (as treated in the previous section).

In case $n = 1$ or $m = 1$ the $r_{i,j}$ have again an easy interpretation: $r_{1,j} = b_{1,j}c_{j,1}$ for $n = 1$ resp. $r_{i,1} = b_{i,1}c_{1,i}$ for $m = 1$. We consider the simplest case $n = m = 1$.

Example 3.3 *Let $\psi(s,t,z) = (s-a_1)(t-d_1)(z-f) + \rho(s,t,z) \in \mathbb{C}[s,t,z]$, where ρ as above. i. e. of degree at most one. Hence*

$$\rho(s,t,z) = -p_1(s-a_1) - q_1(t-d_1) - r_{1,1}(z-f) + k$$

with $p_1, q_1, r_{1,1}, k \in \mathbb{C}$. Then one gets immediately

$$\begin{aligned} V_1 v_1 &= q_1 \\ w_1 W_1 &= p_1 \\ c_{1,1}b_{1,1} &= r_{1,1} \\ c_{1,1}w_1V_1 + v_1 b_{1,1} W_1 &= -k \end{aligned}$$

Hence, by analysing the quadratic equation with roots $c_{1,1}w_1V_1$ and $v_1b_{1,1}W_1$ one gets $\frac{1}{2}(k \pm \sqrt{k^2 - 4p_1q_1r_{1,1}})$ as possible solutions for these products and hence one always finds a matrix $G \in \mathbb{C}^{3\times 3}$ with $\chi_H^{(1,1,1)}(s,t,z) = \psi(s,t,z)$. In the main cases with $q_1, r_{1,1}$ both not zero, one has exactly two normalized solutions, which coincide iff $k^2 - 4p_1q_1r_{1,1} = 0$.

Already the case $(2,1,1)$, i. e. $n = 2, m = 1$ is more involved:

$$\begin{aligned} \chi_G^{(2,1,1)}(s,t,z) \;&=\; (s_1-a_1)(s_2-a_2)(t-d_1)(z-f) \\ &-V_2v_2(s_1-a_1)(t-d_1) - V_1v_1(s_2-a_2)(t-d_1) \\ &-W_1w_1(s_1-a_1)(s_2-a_2) \\ &-b_{2,1}c_{1,2}(s_1-a_1)(z-f) - b_{1,1}c_{1,1}(s_2-a_2)(z-f) \\ &-(c_{1,2}V_2w_1 + b_{2,1}v_2W_1)(s_1-a_1) \\ &-(c_{1,1}V_1w_1 + b_{1,1}v_1W_1)(s_2-a_2) \\ &+(V_2b_{1,1} - V_1b_{2,1})(v_2c_{1,1} - v_1c_{1,2}) \end{aligned}$$

with $s_1 = s_2$. Note, if s_1 and s_2 are algebraically independent indeterminates, then the monomials in $s_1 - a_1, s_2 - a_2, t - d_1, z - f$ occuring on the right hand side above are linearly independent. Therefore their coefficients in $\det(\mathrm{Diag}(s_1, s_2, t, z) - G)$ cannot be expected to be arbitrary, since there are 8 of them and - after normalization- one has only 7 entries of G free. In fact, a straightforward calculation in the Maple package Involutive, cf. [1], yields the following relation for the coefficients $R_{i,j,k,l}$ of $(s_1 - a_1)^i (s_2 - a_2)^j (t - d_1)^k (z - f)^l$ with the above signs in $\det(\mathrm{Diag}(s_1, s_2, t, z) - G)$:

$$
\begin{aligned}
&R_{1,1,0,0}{R_{0,1,1,0}}^2{R_{1,0,0,1}}^2\\
&-2\,R_{1,1,0,0}R_{0,1,1,0}R_{0,0,0,0}R_{1,0,0,1}\\
&-2\,R_{1,1,0,0}R_{1,0,1,0}R_{0,1,1,0}R_{0,1,0,1}R_{1,0,0,1}\\
&+R_{1,1,0,0}{R_{1,0,1,0}}^2{R_{0,1,0,1}}^2 + R_{1,1,0,0}{R_{0,0,0,0}}^2\\
&-2\,R_{1,1,0,0}R_{1,0,1,0}R_{0,0,0,0}R_{0,1,0,1}\\
&+R_{1,0,0,0}R_{1,0,1,0}R_{0,1,0,1}R_{0,1,0,0}\\
&-R_{1,0,0,0}R_{0,0,0,0}R_{0,1,0,0} - R_{1,0,1,0}R_{1,0,0,1}{R_{0,1,0,0}}^2\\
&-R_{0,1,0,1}{R_{1,0,0,0}}^2R_{0,1,1,0}\\
&+R_{1,0,0,1}R_{1,0,0,0}R_{0,1,0,0}R_{0,1,1,0}
\end{aligned}
$$

Checking that this relation is satisfied is an easy substitution in Maple. When passing to $s = s_1 = s_2$ the three monomial $s - a_1, s - a_2, 1$ become dependent:

$$1 = \frac{1}{a_2 - a_1}(s - a_1) - \frac{1}{a_2 - a_1}(s - a_2).$$

Substituting this into the above expression for $\chi_G^{(2,1,1)}(s, t, z)$ yields 7 rather than 8 equations for the entries of G so that one can hope that a solution exists.

Example 3.4 *Let $n = 2, m = 1$ and $\psi(s, t, z) = (s - a_1)(s - a_2)(t - d_1)(z - f) + \rho(s, t, z)$ with*

$$
\begin{aligned}
\rho(s,t,z) \;=\; & -p_1\,(s-a_2)\,(t-d_1)\\
&-p_2\,(s-a_1)\,(t-d_1) - q_1\,(s-a_1)\,(s-a_2)\\
&-r_{1,1}\,(s-a_2)\,(z-f) - r_{2,1}\,(s-a_1)\,(z-f)\\
&-k_1\,(s-a_2) - k_2\,(s-a_1)
\end{aligned}
$$

Assume the most generic case, i. e. $q_1, p_1, p_2, r_{1,1}, r_{2,1}$ are all not zero. Then one can assume that G is normalized, i. e. $v_1 = v_2 = w_1 = 1$ so that $V_1 = p_1, V_2 = p_2, W_1 = q_1$. Also $b_{1,1}c_{1,1} = r_{1,1}, b_{2,1}c_{1,2} = r_{2,1}$. Hence one ends up with just two equations for the remaining unknowns $c_{1,1} \neq 0, c_{1,2} \neq 0$:

$$
\begin{aligned}
k_2 &= (c_{1,2}p_2 + \frac{r_{2,1}}{c_{1,2}}q_1)\\
&\quad - \frac{1}{a_2 - a_1}(p_2\frac{r_{1,1}}{c_{1,1}} - p_1\frac{r_{2,1}}{c_{1,2}})(c_{1,1} - c_{1,2})\\
k_1 &= (c_{1,1}p_1 + \frac{r_{1,1}}{c_{1,1}}q_1)\\
&\quad + \frac{1}{a_2 - a_1}(p_2\frac{r_{1,1}}{c_{1,1}} - p_1\frac{r_{2,1}}{c_{1,2}})(c_{1,1} - c_{1,2})
\end{aligned}
$$

Multiplying both equations with $c_{1,1}c_{1,2}$ and taking resultants, one sees that one has only finitely many solutions for $c_{1,1}, c_{1,2}$, namely roots of certain polynomials of degree 6.
To handle a concrete example, look at

$$G := \left(\begin{array}{cc|c|c} a_1 & 0 & 1 & 1\\ 0 & a_2 & 1 & 1\\ \hline 1 & 1 & d_1 & 1\\ \hline 1 & 1 & 1 & f \end{array}\right)$$

$$
\begin{aligned}
\chi_G^{(2,1,1)}(s,t,z) \;=\; & (s-a_1)\,(s-a_2)\,(t-d_1)\,(z-f)\\
&-(s-a_1)\,(s-a_2) - 2(s-\ a_1)\\
&-(s-a_1)\,(z-f) - (s-a_1)\,(t-d_1)\\
&-2(s-a_2) - (s-a_2)\,(z-f)\\
&-(s-a_2)\,(t-d_1)
\end{aligned}
$$

For the normalized solutions we have $V_1 = V_2 = W_1 = v_1 = v_2 = w_1 = 1$ and $b_{1,1}c_{1,1} = 1, b_{2,1}c_{1,2} = 1$ where $(c_{1,1}, c_{1,2}) = (1, 1)$ as above, or two other solutions with $c_{1,1}^2 + ((a_2 - a_1) - 2 + \frac{4}{a_2 - a_1})c_{1,1} + 1 = 0$ and

$$
\begin{aligned}
c_{1,2} \;=\; & \frac{-c_{1,1}}{2(x-2)}\left(\left(-3\,x+2\,x^2\right){c_{1,1}}^3\right.\\
&+\left(-11\,x^2+19\,x+2\,x^3-12\right){c_{1,1}}^2\\
&+\left(-4\,x^3-29\,x+20+17\,x^2\right)c_{1,1}\\
&\left.-4+11\,x-8\,x^2+2\,x^3\right)
\end{aligned}
$$

with $x := a_2 - a_1$. The case $a_2 - a_1 = 2$ can be handled by first getting an equation for $c_{1,2}$.

Clearly, the case $(n, m) = (3, 1)$ can be dealt with similarly. Since the bivariate case $(n, m) = (2, 2)$ was prepared in the previous section, the generic case for $(2, 2, 1)$ is reduced to a system of algebraic equations in three indeterminates:

Example 3.5 *To find* $G \in \mathbb{C}^{(2+2+1)\times(2+2+1)}$ *with* $\chi_G^{(2,2,1)}(s,t,z) = \psi(s,t,z)$ *for suitable* $\psi(s,t,z) \in \mathbb{C}[s,t,z]$*, concentrate on those solutions where none of the entries of* B, C, V, W, v, w *are equal to zero. Then one can normalize, i. e.* $v = (1,1), w = (1,1)$*,* V, W *are then uniquely determined,*

$$T \begin{pmatrix} A & B \\ C & D \end{pmatrix} T^{-1} = \begin{pmatrix} A & * \\ \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} & D \end{pmatrix}$$

with $A := \mathrm{Diag}(a_1, a_2)$*,* $D := \mathrm{Diag}(d_1, d_2)$*,* $T := \mathrm{Diag}(c_{2,1}c_{1,1}, c_{2,1}c_{1,2}, c_{2,1}, c_{1,1})$*. Then* x *and* $*$ *can be determined from the coefficient* $\psi_z(s,t)$ *of* $z - f$ *in* $\rho(s,t,z)$ *as demonstrated in Proposition 2.1. After that one has to solve algebraic equations for* $c_{1,1}, c_{1,2}$*,* $c_{2,1}$ *only.*

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