Computation of Five and Six Dimensional Bieberbach Groups

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Abstract

This paper is concerned with the computation and classification of 5- and 6-dimensional torsion-free crystallographic groups, also known as *Bieberbach groups*. We describe the basis of an algorithm that decides torsion-freeness of a crystallographic group as well as the triviality of its centre. The computations were done using the computer package CARAT, which handles the enumeration, construction, recognition and comparison problems for crystallographic groups up to dimension 6.

The complete list of isomorphism types of Bieberbach groups up to dimension 6 can be found at http://www-math.math.rwth-aachen.de/~LBFM/carat/.

1 Introduction

A n-dimensional crystallographic group G is a group which contains a normal, torsion-free, maximal abelian subgroup V, of rank n and finite index. Thus an n-dimensional crystallographic group satisfies the short exact sequence

$$0 \longrightarrow V \longrightarrow G \longrightarrow P \longrightarrow 1,$$

where $P \leq \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(V)$ is a finite group acting faithfully on V. The groups P and V are called the *point-group (or holonomy group)* and *translation subgroup* of G, respectively. Crystallographic groups arise as discrete, irreducible subgroups of the group of isometries of the n-dimensional Euclidean space [6]. We say that a crystallographic group G is a *Bieberbach group* if it is torsion-free. Bieberbach groups also appear as fundamental groups of compact, connected, flat Riemannian manifolds (flat manifolds for short). Then X is a flat manifold of dimension n if and only if its fundamental group G is a n-dimensional Bieberbach group. Furthermore, G determines X up to affine equivalence, cf. [6] or [13].

2 The Classification Problem

The problem of classification of isomorphism types of n-dimensional crystallographic groups goes back more than a hundred years ago. Bieberbach's Third Theorem [3] states that for every $n \in \mathbb{N}$, the set of representatives of isomorphism classes of crystallographic (and Bieberbach) groups is finite. The solution for the problem of classification for dimensions 1, 2 and 3 was found in the end of the 19^{th} century, using mainly geometric techniques. In 1948 H. Zassenhaus presented an algorithm [14], based on group-theoretical concepts, that allows the calculation of the set of representatives of isomorphism classes of n-dimensional crystallographic groups for any n.

If G is a n-dimensional crystallographic group, it follows from Bieberbach's Theorems ([2], [3]) that the point-group of G is isomorphic to a finite subgroup of $GL(n, \mathbb{Z})$. Let $\wp_n$ be the set of representatives of conjugacy classes of finite subgroups of $GL(n, \mathbb{Z})$. It follows from the Jordan-Zassenhaus Theorem that the set $\wp_n$ is finite for every n. For instance, for $n = 1, 2, 3, 4, 5, 6$, the set $\wp_n$ has 2, 13, 73, 710, 6079 and 85311 elements, respectively (see [9]).

Once given the set $\wp_n$ of representatives of conjugacy classes of finite subgroups of $GL(n, \mathbb{Z})$ and the generators of the normalizer in $GL(n, \mathbb{Z})$ of each element of $\wp_n$, Zassenhaus' algorithm allows one to calculate the set of representatives of isomorphism classes of n-dimensional crystallographic groups. In [4] one can find the list of crystallographic groups for dimensions ≤ 4. Furthermore, in [9] the numbers for dimensions 5 and 6 were added (see Table 1).

We would like to notice that $n = 4$ seems to be the last dimension where a *complete list* of isomorphism classes of n-dimensional crystallographic groups still makes sense, since the increase in the number of isomorphism classes for higher dimensions does not allow a complete list in a *readable form*. An alternative kind of classification for crystallographic group up to dimension 6 is suggested in [9].

For every finite group P, Auslander and Kuranishi [1] have shown that there is a Bieberbach group having point-group isomorphic to P. Following the terminology in Hiller and Sah [7], we will call a finite group P *primitive* if it can be realized as point-group of a Bieberbach group G

with finite commutator quotient. In contrast to the result of Auslander and Kuranishi, not every finite group is primitive. In [7], Hiller and Sah prove that a finite group P is primitive if and only if no cyclic Sylow p-subgroup of P has a normal complement. Let X be a flat manifold that has G as its fundamental group. Then it is well known that the first Betti number of the manifold X is zero if and only if the commutator quotient of G is finite, which is also equivalent to the triviality of the centre of G [7]. Due to the *Calabi construction* [5], n-dimensional Bieberbach groups with trivial centre have a relevant role in the classification of n-dimensional Bieberbach groups in general.

3 Deciding Torsion-freeness

Even though the problem of classification of isomorphism types of crystallographic groups dates back to the 19^{th} century, the idea to study and classify the torsion-free ones in particular came much later, when the study of flat manifolds was initiated. In [4] one can find the list of Bieberbach groups up to dimension 4. For $n = 1, 2, 3, 4$, there are 1, 2, 10 and 74 n-dimensional Bieberbach groups, respectively. Below we describe the basis of an algorithm that decides torsion-freeness of a crystallographic group and verifies if it has trivial centre.

The description of a crystallographic group G is based on the following property: If $x_1, x_2, \ldots, x_k$ are the generators of the point-group $P \leq GL(n, \mathbb{Z})$ of G, then a generating set of G can be given in the form

$$\left\{ \begin{pmatrix} x_1 & v_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_k & v_k \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} Id & v \\ 0 & 1 \end{pmatrix}, v \in \mathbb{Z}^n \right\},$$

where $v_1, \ldots, v_k \in \mathbb{Q}^n$ and Id is the $n \times n$ identity matrix. We denote $\begin{pmatrix} x_i & v_i \\ 0 & 1 \end{pmatrix}$ by g_i.

The following lemma gives the condition that a crystallographic group G must satisfy in order to be torsion-free.

Lemma 3.1 *Let V be a torsion-free $\mathbb{Z}P$-module, and G the extension of P by V corresponding to $\alpha \in H^2(P, V)$. Then G is torsion-free if and only if $\mathrm{res}^P_H(\alpha) \neq 0$ for every cyclic subgroup H of prime order of P.*

Let $\phi : G \to P \cong G/V$ be the natural homomorphism. Let $g \in G$ be an element of finite order. Without loss of generality, we can assume that $g^p = 1$ for some prime p. And since V is torsion-free, $\phi(g)$ is an element of order p in P, and $\mathrm{res}^P_H(\alpha) = 0$, where $H =< \phi(g) >$. Conversely, assume that $\mathrm{res}^P_H = 0$. Then $\phi^{-1}(H)$ splits and therefore G has an element of finite order.

Remark 3.2 *Let V, G, P and α be as given in Lemma 3.1, and define $t(n, x) := 1 + \cdots + x^{n-1}$. Let $g \in G$ given by $g = \begin{pmatrix} h & \nu \\ 0 & 1 \end{pmatrix}$, where $\phi(g) = h \in P$ has order m. Then $\mathrm{res}^P_{<h>}\alpha = 0$ if and only if $t(m, h).\nu \in t(m, h).V$.*

A trivial calculation shows that one needs only to do such tests for representatives of conjugacy classes of cyclic subgroups of prime order of P.

In short, the main points of the algorithm are :

1) Given the generators $g_1, g_2, \ldots, g_k$ of G, obtain coset representatives of V in G corresponding to the *point-group* P (via ϕ);
2) In view of Lemma 3.1, restrict the list from 1) to those elements giving rise to representatives of conjugacy classes of elements of prime order in P;
3) For every element $g \in G$ from the list 2), test if $t(p, \phi(g)).v_g \in t(p, \phi(g)).V$, where v_g is the translation component of g and p is a prime such that $\phi(g)^p = e$. If so, the solution of this $\mathbb{Z}$-linear system will give an element of order p in G. Otherwise, G is torsion-free.

The centre of a space group in general is the lattice fixed by P, and it is easy to determine via solving $(\phi(g_i) - 1).v = 0$, for $i = 1, \ldots, k$. Note that this calculation has to be done only once for all groups in a specific $\mathbb{Q}$-class.

The algorithm has been implemented in C and uses as input the results presented in [9]. The computer programs and data which were used to obtain these results are part of the package CARAT, which is also available at the above address. The package, the algorithms, a lot of the underlying theory are described in [8] and [9].

As a result, we have the following table for n-dimensional Crystallographic and Bieberbach groups, $n \leq 6$.

dimension	1	2	3	4	5	6
n° of crystallographic groups	2	17	219	4783	222.018	28.927.922
n° of Bieberbach groups	1	2	10	74	1060	38746

Table 1

4 Five dimensional Bieberbach groups

Through our calculations, we have obtained **1060** Bieberbach groups in dimension 5, of which 101 have trivial centre. On working first with those 5-dimensional Bieberbach groups with trivial centre, we classified the possible isomorphism types of point-groups.

Let $(C_n)^k$ denote the direct product of k copies of the cyclic group of order n, D_n the dihedral group of order n, S_n and A_n the symmetric and alternating groups, and Γ the group of order 16 and nilpotency class 2, isomorphic to $(C_2 \times C_2) \rtimes C_4$.

Theorem 4.1 *A finite group P can be realized as point-group of a 5-dimensional Bieberbach group with trivial centre if and only if it is isomorphic to $(C_2)^k$, for $2 \leq k \leq 4$, $C_2 \times C_4$, D_{12}, D_8, $D_8 \times C_2$ or Γ.*

The above theorem shows that Theorem 1 of [11] is not correct, where the group $C_4 \times C_2 \times C_2$ is included in the list of finite groups. The given example of a Bieberbach group with such point-group and trivial centre is in fact not torsion-free (for instance, the element $c_2.b_2.a_2$ has order 4).

By considering all 1060 groups, the theorem below, that lists the finite groups that can be realized as the point-group of a 5-dimensional Bieberbach group, agrees with Theorem 1 of [12].

Theorem 4.2 *A finite group $P \neq \{e\}$ can be realized as point-group of a 5-dimensional Bieberbach group if and only if it is isomorphic to C_n, for $2 \leq n \leq 6$, $n = 8, 10, 12$, $(C_2)^k$, for $2 \leq k \leq 4$, $C_2 \times C_4$, $C_4 \times C_2 \times C_2$, $C_3 \times C_3$, $C_6 \times C_2$, $C_6 \times C_3$, $C_6 \times C_2 \times C_2$, $C_{12} \times C_2$, S_3, D_8, D_{12}, $D_8 \times C_2$, $S_3 \times C_3$, $D_{12} \times C_2$, A_4, $A_4 \times C_2$, $A_4 \times C_2 \times C_2$, S_4 or Γ.*

Following the terminology used in [8] and [9], we have the following tables of 5-dimensional Bieberbach groups.

family symbol	Isom. type point-group	no. $\mathbb{Q}$-classes	no. $\mathbb{Z}$-classes	no. aff. classes
$1,1,1;1;1$	$C_2 \times C_2$	1	1	1
$1,1;1,1;1$	$C_2 \times C_2$	1	2	2
$1,1;1;1;1$	$C_2 \times C_2 \times C_2$	2	3	17
$1;1;1;1;1$	$C_2 \times C_2 \times C_2, C_2 \times C_2 \times C_2 \times C_2$	3	8	44
$2-1,2-1;1$	D_8	1	1	1
$2-1;1,1;1$	D_8	2	3	3
$2-1;1;1;1$	$C_2 \times C_4, C_2 \times D_8, D_8$	6	13	25
$2-1;2-1;1$	Γ	2	2	4
$2-2;1;1;1$	$C_2 \times S_3$	2	4	4
Σ		**20**	**37**	**101**

Table 2 - 5-dimensional Bieberbach groups with trivial centre (ie. first Betti no. = 0)

family symbol	Isom. type point-group	no. $\mathbb{Q}$-classes	no. $\mathbb{Z}$-classes	no. aff. classes
$1,1,1,1,1$	C_1	1	1	1
$1,1,1,1;1$	C_2	2	3	3
$1,1,1;1,1$	C_2	2	5	5
$1,1,1;1;1$	C_2^2	2	8	21
$1,1;1,1;1$	C_2^2	2	9	31
$1,1;1;1;1$	C_2^2, C_2^3	4	43	236
$1;1;1;1;1$	C_2^3, C_2^4	3	22	290
$2-1',2-1';1$	C_4	1	1	1
$2-1;1,1,1$	C_4	1	2	2
$2-1;1,1;1$	$C_2 \times C_4, C_4, D_8$	8	36	68
$2-1;1;1;1$	$C_2^2 \times C_4, C_2 \times C_4, C_2 \times D_8, D_8$	8	62	179
$2-1;2-1;1$	$C_2 \times C_4, \Gamma$	2	5	12
$2-2',2-2';1$	$C_2 \times C_3, C_3$	2	2	2
$2-2;1,1,1$	$C_2 \times C_3, C_3$	2	3	3
$2-2;1,1;1$	$C_2^2 \times C_3, C_2 \times C_3, C_2 \times S_3, S_3$	10	27	31
$2-2;1;1;1$	$C_2^3 \times C_3, C_2^2 \times C_3, C_2^2 \times S_3, C_2 \times S_3$	7	12	42
$2-2;2-1;1$	$C_2 \times C_3 \times C_4, C_3 \times C_4$	3	3	3
$2-2;2-2;1$	$C_2 \times C_3, C_2 \times C_3^2, C_3^2, C_3 \times S_3$	4	8	10
$3;1,1$	$A_4, C_2 \times A_4$	2	4	4
$3;1;1$	$C_2 \times A_4, C_2^2 \times A_4, S_4$	5	9	11
$4-1';1$	C_8	1	1	1
$4-2';1$	$C_3 \times C_4$	1	1	1
$4-3';1$	$C_2 \times C_5, C_5$	2	2	2
Σ		**75**	**269**	**959**

Table 3 - 5-dimensional Bieberbach groups with non-trivial center

5 Six dimensional Bieberbach groups

The following tables give a classification of 6-dimensional Bieberbach groups. As to the isomorphim types of point groups, the notation $[n,k]$ refers to the classification of solvable groups with small order given in GAP (see [10]) and n stands for the order of the group in question. Note that in the notation $C_3 \rtimes C_4, C_3 \rtimes C_8$ and $C_3^2 \rtimes C_2$ the top C_2 acts by inverting, in the latter case on both components.

family symbol	Isom. type point-group	no. $\mathbb{Q}$-classes	no. $\mathbb{Z}$-classes	no. aff. classes
$1,1,1,1;1;1$	C_2^2	1	1	1
$1,1,1;1,1;1$	C_2^2	1	2	2
$1,1,1;1;1;1$	C_2^3	2	3	18
$1,1;1,1;1,1$	C_2^2	1	4	4
$1,1;1,1;1;1$	C_2^3	2	7	62
$1,1;1;1;1;1$	$C_2^i,\ 3 \leq i \leq 4$	5	36	791
$1;1;1;1;1;1$	$C_2^i,\ 3 \leq i \leq 5$	6	49	2727
$2-1,2-1;1,1$	D_8	1	1	1
$2-1,2-1;1;1$	$C_2 \times D_8, D_8$	2	7	11
$2-1;1,1,1;1$	D_8	2	3	3
$2-1;1,1;1,1$	D_8	1	2	3
$2-1;1,1;1;1$	$C_2 \times C_4, C_2 \times D_8, D_8$	15	67	173
$2-1;1;1;1;1$	$C_2^2 \times C_4, C_2^2 \times D_8, C_2 \times D_8$	16	113	883
$2-1;2-1;1,1$	$C_2 \times D_8, \Gamma$	3	7	9
$2-1;2-1;1;1$	$C_2 \times C_4, C_2 \times D_8, C_2 \times \Gamma, \Gamma,$ [16, 10], [32, 33], [32, 36]	20	74	197
$2-1;2-1;2-1$	Γ, [32, 33]	2	4	5
$2-2;1,1;1;1$	$C_2^2 \times S_3, C_2 \times S_3$	6	13	13
$2-2;1;1;1;1$	$C_2^2 \times S_3$	6	13	71
$2-2;2-1;1;1$	D_{24}, [24, 11]	4	10	10
$3;1;1;1$	$C_2 \times S_4$	3	3	4
$3;2-2;1$	$C_2 \times S_4, S_4$	2	9	9
$4-1';1;1$	D_{16}	1	1	1
$4-1;1;1$	[16, 13], [16, 8], [32, 47]	3	3	3
$4-1;2-1$	[32, 46], [64, 250]	3	3	3
Σ		**108**	**435**	**5004**

Table 4 - 6-dimensional Bieberbach groups with trivial centre (ie. first Betti no. = 0)

family symbol	Isom. type point-group	no. $\mathbb{Q}$-classes	no. $\mathbb{Z}$-classes	no. aff. classes
$1,1,1,1,1,1$	C_1	1	1	1
$1,1,1,1,1;1$	C_2	2	3	3
$1,1,1,1;1,1$	C_2	2	5	5
$1,1,1,1;1;1$	C_2^2	2	8	22
$1,1,1;1,1,1$	C_2	1	3	3
$1,1,1;1,1;1$	C_2^2	3	21	87
$1,1,1;1;1;1$	$C_2^i, 2 \leq i \leq 3$	4	55	440
$1,1;1,1;1,1$	C_2^2	1	9	29
$1,1;1,1;1;1$	$C_2^i, 2 \leq i \leq 3$	4	98	1078
$1,1;1;1;1;1$	$C_2^i, 3 \leq i \leq 4$	7	214	8403
$1;1;1;1;1;1$	$C_2^i, 3 \leq i \leq 5$	5	158	13839
$2-1', 2-1';1,1$	C_4	1	2	2
$2-1', 2-1';1;1$	$C_2 \times C_4, C_4$	2	6	8
$2-1, 2-1;1;1$	D_8	2	14	25
$2-1;1,1,1,1$	C_4	1	2	2
$2-1;1,1,1;1$	$C_2 \times C_4, C_4, D_8$	8	37	89
$2-1;1,1;1,1$	$C_2 \times C_4, C_4, D_8$	4	38	88
$2-1;1,1;1;1$	$C_2^2 \times C_4, C_2 \times C_4, C_2 \times D_8, D_8$	21	392	2187
$2-1;1;1;1;1$	$C_2^3 \times C_4, C_2^2 \times C_4, C_2^2 \times D_8, C_2 \times C_4, C_2 \times D_8, D_8$	18	525	4972
$2-1;2-1;1,1$	$C_2 \times C_4, C_4^2, [16,10], \Gamma$	4	24	71
$2-1;2-1;1;1$	$C_2^2 \times C_4, C_2 \times C_4, C_2 \times D_8, C_2 \times \Gamma, C_4^2,$ $C_4 \times D_8, [16,10], \Gamma, [32,33], [32,36]$	20	173	757
$2-2', 2-2';1,1$	$C_2 \times C_3, C_3$	2	3	3
$2-2', 2-2';1;1$	$C_2^2 \times C_3, C_2 \times C_3$	3	4	4
$2-2, 2-2;1;1$	$C_2 \times S_3, S_3$	2	7	7
$2-2;1,1,1,1$	$C_2 \times C_3, C_3$	2	3	3
$2-2;1,1,1;1$	$C_2^2 \times C_3, C_2 \times C_3, C_2 \times S_3, S_3$	10	27	33
$2-2;1,1;1,1$	$C_2^2 \times C_3, C_2 \times C_3, C_2 \times S_3, S_3$	5	20	26
$2-2;1,1;1;1$	$C_2^3 \times C_3, C_2^2 \times C_3, C_2^2 \times S_3, C_2 \times S_3$	18	118	391
$2-2;1;1;1;1$	$C_2^i \times C_3, 2 \leq i \leq 4, C_2^i \times S_3, 1 \leq i \leq 3$	15	107	652
$2-2;2-1;1,1$	$C_2 \times C_3 \times C_4, C_3 \times C_4$	3	8	10
$2-2;2-1;1;1$	$(C_3 \rtimes C_4), C_2 \times (C_3 \rtimes C_4), C_2^i \times C_3 \times C_4, 0 \leq i \leq 2,$ $C_2 \times C_3 \times D_8, C_2 \times C_4 \times S_3, C_2 \times D_{24},$ $C_3 \times D_8, C_4 \times S_3, D_{24}, [24,11]$	26	98	140
$2-2;2-2;1,1$	$C_2^i \times C_3^j, 1 \leq i,j \leq 2, C_2 \times C_3 \times S_3, C_3^2, C_3 \times S_3$	8	21	32
$2-2;2-2;1;1$	$C_3^2 \rtimes C_2, C_2 \times (C_3^2 \rtimes C_2), C_2^i \times C_3^j, 1 \leq i,j \leq 2,$ $C_2 \times C_3 \times S_3, C_2 \times S_3, C_3 \times S_3, S_3 \times S_3$	20	65	89
$3;1,1,1$	$A_4, C_2 \times A_4$	2	4	4
$3;1,1;1$	$C_2 \times A_4, C_2^2 \times A_4, C_2 \times S_4, S_4$	11	37	47
$3;1;1;1$	$C_2^2 \times A_4, C_2^3 \times A_4, C_2 \times S_4$	8	18	57
$3;2-1;1$	$C_2 \times C_4 \times A_4, C_4 \times A_4$	3	5	7
$3;2-2;1$	$A_4, C_2 \times A_4, C_2^2 \times A_4, C_2 \times C_3 \times A_4, C_3 \times A_4, C_3 \times S_4$	9	30	55
$4-1';1,1$	C_8	1	2	2
$4-1';1;1$	$C_2 \times C_8, C_8, D_{16}$	3	8	11
$4-1;1;1$	$(C_3 \rtimes C_8), C_3 \times Q_8, Q_8, [16,11], [16,13], [16,8], [32,31]$	8	15	23
$4-2';1,1$	$C_3 \times C_4$	1	1	1
$4-2';1;1$	$C_3 \times C_4, D_{24}$	2	3	3
$4-2;1;1$	$C_3 \rtimes C_4, C_2 \times C_3 \times S_3, C_3 \times (C_3 \rtimes C_4), C_3 \times D_8, C_3 \times S_3$	5	10	12
$4-3';1,1$	$C_2 \times C_5, C_5$	2	3	3
$4-3';1;1$	$C_2^2 \times C_5, C_2 \times C_5, C_2 \times D_{10}, D_{10}$	5	8	8
$5-1;1$	$C_2 \times [80,52], [80,52]$	2	3	8
Σ		**289**	**2416**	**33742**

Table 5 - 6-dimensional Bieberbach groups with non-trivial centre

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