# Vessiot's Equivalence Method Applied to Linear Partial Differential Operators

Arne Lorenz

Lehrstuhl B für Mathematik
RWTH Aachen

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# Outline

# Motivation

## Example

Linear partial differential operators (LPDOs) of order 2:

$$L = D_{x^1} D_{x^2} + a\, D_{x^1} + b\, D_{x^2} + c$$

under gauge transformations:

$$L \mapsto g^{-1} L g, \qquad g = g(x^1, x^2).$$

Questions:

- Equivalence: $L$, $L'$ $\quad \rightsquigarrow \quad$ $g^{-1} L g = L'$?
- Invariants?

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- Equivalence: $L$, $L'$ $\quad \rightsquigarrow \quad g^{-1} L g = L'$?
- Invariants? Laplace:

$$h = c - a_{x^1} - ab, \qquad k = c - b_{x^2} - ab.$$

- Generating set of invariants?
- Larger examples?

# Outline

# Natural bundles

Let $X$ be a manifold, coordinates $(x) = (x^1, \dots, x^n)$.

- $\mathrm{Diff}_{\mathrm{loc}}(X, X)$: local diffeomorphisms $\varphi : X \to X$.
- A natural bundle is a fibre bundle

$$\pi : \mathcal{F} \to X : (x, u) \to (x)$$

  such that each $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(X, X)$ continues to $\tilde{\varphi} : \mathcal{F} \to \mathcal{F}$.
- A section $\omega : X \to \mathcal{F} : (x) \mapsto (x, u = \omega(x))$ is called geometric object.

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- $\varphi$ is an equivalence between $\omega$, $\gamma : X \to \mathcal{F} \Leftrightarrow$

$$\omega \circ \tilde{\varphi} = \gamma$$

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- $\psi : \mathcal{F} \to \mathbb{R}$ is an invariant if $\psi \circ \tilde{\varphi} = \psi \quad \forall \varphi \in \mathrm{Diff}_{\mathrm{loc}}(X, X)$.

# Symmetry Groupoids

- The jet groupoid $\Pi_q = \Pi_q(X, X)$ has coordinates:

$$(x, y, y_x, \ldots, y_q).$$

- Like $\text{Diff}_{\text{loc}}(X, X)$, $\Pi_q$ acts on $\mathcal{F}$.
- Symmetry groupoid $\mathcal{R}_q(\omega)$ of $\omega : X \to \mathcal{F}$ defined by:

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- Prolongation

$$0 \longrightarrow \mathcal{R}_{q+1}(\omega) \longrightarrow \Pi_{q+1} \underset{j_1(\omega)}{\overset{j_1(\Phi_\omega)}{\rightrightarrows}} J_1(\mathcal{F})$$

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0 & \longrightarrow & \mathcal{R}_q^{(1)}(\omega) & \longrightarrow & \Pi_q & \rightrightarrows & \mathcal{F}_1
\end{array}$$

Question: Is $\mathcal{R}_q^{(1)}(\omega) = \mathcal{R}_q(\omega)$?

# Integrability Conditions

## Theorem

*$\pi_q^{q+1} : \mathcal{R}_{q+1}(\omega) \to \mathcal{R}_q(\omega)$ is surjective if and only if there is a section $c : \mathcal{F} \to \mathcal{F}_1$:*

- *$c$ is equivariant:*

$$c(af_q) = c(a)f_q \qquad \forall f_q \in \Pi_q,\ a \in \mathcal{F}$$

- *$c$ fulfills the Vessiot structure equations:*

$$(I \circ j_1)(\omega) = c(\omega).$$

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If in addition the symbol of $\mathcal{R}_q(\omega)$ is 2-acyclic, $\mathcal{R}_q(\omega)$ is integrable, i. e.:

$$\pi^{q+r+s}_{q+r} : \mathcal{R}_{q+r+s}(\omega) \to \mathcal{R}_{q+r}(\omega) \qquad \forall r, s \in \mathbb{N}$$

are all surjective.

# Equivalence

## Theorem

*Two geometric objects $\omega$ and $\gamma$ on $\mathcal{F}$ are equivalent if:*

- *All invariants $\psi$ on $\mathcal{F}$ coincide for some $x, y \in X$:*

$$\psi(\omega(y)) = \psi(\gamma(x)),$$

- *$\mathcal{R}_q(\omega)$, $\mathcal{R}_q(\gamma)$ are integrable with $c : \mathcal{F} \to \mathcal{F}_1$:*

$$\begin{aligned}(I \circ j_1)(\omega) &= c(\omega),\\ (I \circ j_1)(\gamma) &= c(\gamma).\end{aligned}$$

## Example

In the introductory example $(\omega, \Omega)$ is equivalent to $(\bar{\omega}, \bar{\Omega})$ if:

$$d\omega = c_1\Omega, \quad d\bar{\omega} = c_1\bar{\Omega}$$

# Outline

# Groupoids $\Theta_q$ of Gauge Transformations

- A simple observation for LPDOs:

$$\left(g^{-1}Lg\right)u(x) = g^{-1}L\left(gu(x)\right)$$

$\rightsquigarrow$ Find a groupoid containing only transformations:

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- On $Y = X \times \mathbb{R}$, coordinates $(x, u)$, $(y, v)$ define $\Theta_q \leq \Pi_q(Y, Y)$:

$$y = x, \quad u\, v_u = v.$$

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- All previous slides remain true if $\Pi_q$ is restricted to $\Theta_q$:

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Theta_q \overset{\Phi_\omega}{\underset{\omega}{\rightrightarrows}} \mathcal{F}$$

- Natural $\Theta_q$-bundles: $\Theta_q$-action on $\mathcal{F}$.

# Laplace Example – Natural Bundle

- Construct a natural $\Theta_q$-bundle for:

$$L = D_{x^1} D_{x^2} + a D_{x^1} + b D_{x^2} + c.$$

- Gauge transformation:

$$\begin{aligned} g^{-1} L g &= D_{x^1} D_{x^2} + (\frac{g_{x^2}}{g} + a) D_{x^1} + (\frac{g_{x^1}}{g} + b) D_{x^2} \\ &\quad + \frac{g_{x^1 x^2}}{g} + a \frac{g_{x^1}}{g} + b \frac{g_{x^2}}{g} + c \end{aligned}$$

- Reminder: $\Theta_q$ : $\quad y = x, \quad u\, v_u = v$.
  $\Rightarrow v(x, u) = g(x) u$.
- Natural $\Theta_q$-bundle $\mathcal{F} = Y \times \mathbb{R}^3$, coordinates $(x, u, a, b, c)$:

$$\begin{aligned} a &= (\frac{v_{x^2}}{v} + \hat{a}), \\ b &= (\frac{v_{x^1}}{v} + \hat{b}), \\ c &= \frac{v_{x^1 x^2}}{v} + \hat{a} \frac{v_{x^1}}{v} + \hat{b} \frac{v_{x^2}}{v} + \hat{c}. \end{aligned}$$

# Laplace Example – Prolongation

$$L = D_{x^1} D_{x^2} + a D_{x^1} + b D_{x^2} + c$$

- $\Theta_q$-action on $\mathcal{F} = Y \times \mathbb{R}^3$:

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- Coordinates of $J_1(\mathcal{F})$:

$$a_{x^1}, \quad a_{x^2}, \quad a_u, \quad b_{x^1}, \quad b_{x^2}, \quad b_u, \quad c_{x^1}, \quad c_{x^2}, \quad c_u.$$

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- Coordinates of $J_{1,X}(\mathcal{F})$:

  $$a_{x^1}, \quad a_{x^2}, \quad b_{x^1}, \quad b_{x^2}, \quad c_{x^1}, \quad c_{x^2}.$$

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### Theorem

*If the $\Theta_q$-action on $\mathcal{F}$ depends on $v_{x^\mu}/v$ only then*

- *Sections $\omega(x)$ of $\mathcal{F}$ are well-defined.*
- *$j_1(\omega)(x)$ restricts to the $\Theta_q$-subbundle $J_{1,X}(\mathcal{F}) \subseteq J_1(\mathcal{F})$.*

# Invariants on $\Theta_q$-bundles

- $\Theta_q$ is defined by:

$$y = x, \quad u\, v_u = v.$$

- Invariant $\psi : J_{r,X}(\mathcal{F}) \to \mathbb{R}$.
- Derivatives $D_{x^i}\psi$ are invariants on $J_{r+1,X}(\mathcal{F})$.

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- If all coordinates of $\mathcal{F}_1$ are invariants,

$$\mathcal{F}_1 \cong \mathcal{F} \times \mathbb{R}^k$$

$\Rightarrow$ the invariants on $\mathcal{F}_1$ are a generating set.

- Next bundle of integrability conditions $\mathcal{F}_2$:

$$\mathcal{F}_2 \cong \mathcal{F}_1 \times \mathbb{R}^{nk}$$

# Laplace Example – Vessiot Structure Equations

$$L = D_{x^1} D_{x^2} + a D_{x^1} + b D_{x^2} + c$$

- $\Theta_q$-action on $\mathcal{F} = Y \times \mathbb{R}^3$:

$$a = (\frac{v_{x2}}{v} + \hat{a}), \quad b = (\frac{v_{x1}}{v} + \hat{b}), \quad c = \frac{v_{x^1 x^2}}{v} + \hat{a}\,\frac{v_{x1}}{v} + \hat{b}\,\frac{v_{x2}}{v} + \hat{c}$$

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- Coordinates of $\mathcal{F}_1$:

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- Better coordinates of $\mathcal{F}_1$:

$$h = c - a_{x^1} - ab, \quad a_{x^2}, \quad b_{x^1}, \quad k = c - b_{x^2} - ab.$$

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- No equivariant sections $\mathcal{F} \to \mathcal{F}_1$.
- Vessiot structure equations on $\mathcal{F}_2 \cong \mathcal{F}_1 \times \mathbb{R}^4$:

$$h_{x^1} = c_1(h,k), \;\; h_{x^2} = c_2(h,k), \;\; k_{x^1} = c_3(h,k), \;\; k_{x^2} = c_4(h,k).$$

$\Rightarrow \{h, k\}$ is a generating set of invariants.

# Example from Shemyakova & Winkler [SW07]

## Example

Third order LPDOs under gauge transformations:

$$(D_x + qD_y)D_{xy} + a_{20}D_{xx} + a_{11}D_{xy} + a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00}$$

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Equivalence conditions: $\rightsquigarrow$ second order invariants.

| Order | Σ | Invariants |
|---|---|---|
| 0 | 2 | $q, \quad I_1 = 2a_{20}q^2 - a_{11}q + 2a_{02}$ |
| 1 | 8 | $q_x, q_y, I_x^1, I_y^1$<br>4 new invariants |
| 2 | 15 | 14 = 6 + 8 old, 1 new |
| 3 | 21 | 21 old |

Generating set from [SW07] is very compact!

# The end.

Thanks!

# Literature


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