PREREQUISITES TO AACHEN SUMMER SCHOOL

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Abstract. This is preparatory material, to be thought about before the beginning of the summer school. No deep study is suggested, but moderate familiarity with the discussed concepts will be expected during the talks.

1. Algebra

1.1. **Categorical concepts.** A *category* $\mathcal{C}$ consists of *objects* A, B, C,..., and of sets of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ subject to some reasonable conditions:

(1) for each object A, $\mathrm{Hom}_{\mathcal{C}}(A, A)$ contains a distinguished element, the *identity morphism* $\mathrm{id}_A : A \longrightarrow A$;

(2) for any pair $\phi: A \longrightarrow B$ and $\psi: B \longrightarrow C$ of morphisms in $\mathcal{C}$, $\psi \circ \phi : A \longrightarrow B \longrightarrow C$ is also a morphism in $\mathcal{C}$;

(3) for all $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\mathrm{id}_B \circ \phi = \phi \circ \mathrm{id}_A = \phi$;

(4) for all $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and all $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and all $\rho \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, $\rho \circ (\psi \circ \phi) = (\rho \circ \psi) \circ \phi$.

We usually consider a ring R and the category of left R-modules. In this category R-mods, all morphisms are *R-linear*: $\phi(rm) = r\phi(m)$ for all $r \in R$, $m \in M \in R$-mods.

A (covariant) *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a map on the objects

$$\mathcal{C} \ni A \mapsto \mathrm{F}(A) \in \mathcal{D},$$

and a map on the morphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \ni \phi \mapsto \mathrm{F}(\phi) \in \mathrm{Hom}_{\mathcal{D}}(\mathrm{F}(A), \mathrm{F}(B))$$

with reasonable properties: $\mathrm{F}(\psi) \circ \mathrm{F}(\phi) = \mathrm{F}(\psi \circ \phi)$ for all pairs of morphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$.

An *additive category* is one where each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an Abelian group, and where each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ induces a group homomorphism $\mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$ by precomposition with ϕ, and where each $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ induces a group homomorphism $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$ by postcomposition with ϕ. (Note: that ϕ and ψ induce such maps follows from the category axioms above; the interesting bit is that these induced maps are homomorphisms of groups.) Moreover, an additive category is required to have a *zero object* 0 which allows precisely the zero group of morphisms to and from it, and for each pair of objects A, B there must be a a *product* $A \times B$ (which allows for a bijection of $\mathrm{Hom}_{\mathcal{C}}(A \times B, C)$ with $\mathrm{Hom}_{\mathcal{C}}(A, C) \times \mathrm{Hom}_{\mathcal{C}}(B, C)$ that is preserved under composition with $C \longrightarrow C'$). Note that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ contains a special elemtn, called *zero*, given by $A \longrightarrow 0 \longrightarrow B$.

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An *additive functor* (between additive categories) is one for which the association

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathrm{F}(A),\mathrm{F}(B))$$

is a group homomorphism.

Suppose now that $\mathcal{C}$ is additive. If $\phi\colon A \longrightarrow B$ is a morphism in $\mathcal{C}$ then a *kernel* of ϕ is a morphism $\ker(\phi)\colon A' \longrightarrow A$ such that $\phi \circ \ker(\phi) = 0$ and such that any other map $\psi\colon A'' \longrightarrow A$ with $\phi \circ \psi = 0$ factors through A'. A *cokernel* of ϕ is a map $\mathrm{coker}(\phi)\colon B \longrightarrow B'$ with $\mathrm{coker}(\phi) \circ \phi = 0$ and every other map $\psi\colon B \longrightarrow B''$ with $\psi \circ \phi = 0$ factors through B'.

An *Abelian category* is an additive category where each morphism has both kernel and cokernel, and where the following two technical conditions holds. If $\phi\colon A \longrightarrow B$ is such that the only maps $\psi\colon A' \longrightarrow A$ with $\phi \circ \psi = 0$ are the zero maps from A' to A (one could say that "ϕ behaves like a monomorphism") then $\phi = \ker(\mathrm{coker}(\phi))$ (i.e., maps that behave like monomorphisms all arise as kernels). A similar condition is required for cokernels and "maps like epimorphisms". (State this condition and prove that each kernel behaves like a monomorphism and each cokernel behaves like an epimorphism!)

There is a grand theorem that says that any Abelian category can be identified with a subcategory of the category of Abelian groups. In particular, one may pretend that objects of an Abelian $\mathcal{C}$ are Abelian groups and have *elements.* (And we will do that.)

Warning: while direct sums and products exist in Ab, the category of Abelian groups, the embedding theorem does not guarantee this for all Abelian categories (the product in Ab may not be in the image of the embedding.)

In an Abelian category $\mathcal{C}$ one may hence speak of the *image* of a map $\phi : A \longrightarrow B$ by $\mathrm{im}(\phi) = \{\phi(a) \mid a \in A\}$. By a basic isomorphism theorem, (and abuse of notation) $\mathrm{im}(\phi) \cong A/\ker(\phi)$, the quotient of groups. (Explain where notation was abused.)

A *complex* $A^\bullet$ in an Abelian category is a sequence

$$\cdots A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

where each composition $d^i \circ d^{i-1}$ is *zero.* This means that the morphism $d^i \circ d^{i-1}\colon A^i \longrightarrow A^{i+2}$ has kernel equal to A^i. The index i on the object A^i is the *cohomological degree.*

The *i-th cohomology of a complex* $A^\bullet$ is the quotient $H^i(A^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$. It is an Abelian group measuring how far $A^\bullet$ is from being *exact*, which is the case if and only if $\ker(d^i) = \mathrm{im}(d^{i-1})$ for all i. An exact complex with 3 terms is a *short exact sequence.*

A functor $\mathrm{F}\colon \mathcal{C} \longrightarrow \mathcal{D}$ between Abelian categories is *exact* if applying it to short exact sequences produces short exact sequences. A standard example is to take $\mathcal{C}$ the category of Abelian groups and group homomorphisms, and $\mathcal{D}$ the category of $\mathbb{Q}$-vector spaces and maps of these, and to let F be the tensor product $(-) \otimes_{\mathbb{Z}} \mathbb{Q}$. (Exercise: let F be the functor $(-) \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$ for $p > 0$ sending Abelian groups to modules over the ring $\mathbb{Z}/p\mathbb{Z}$ and show by example that this is not exact.)

If $\mathcal{C}$ is an Abelian category, then the complexes in $\mathcal{C}$ form another category. Let $A^\bullet$ and $B^\bullet$ be two complexes in $\mathcal{C}$ with maps $\alpha^i : A^i \longrightarrow A^{i+1}$ and $\beta^i : B^i \longrightarrow B^{i+1}$. A morphism from $A^\bullet$ to $B^\bullet$ in the *complex category* of $\mathcal{C}$ is a choice of morphisms

$\phi^i : A^i \longrightarrow B^i$ for each i such that all the following diagrams "commute":

$$\begin{array}{ccc} A^i & \xrightarrow{\phi^i} & B^i \\ {\scriptstyle \alpha^i}\downarrow & & \downarrow{\scriptstyle \beta^i} \\ A^{i+1} & \xrightarrow[\phi^{i+1}]{} & B^{i+1} \end{array}$$

In other words, we require $\beta^i \circ \phi^i = \phi^{i+1} \circ \alpha^i$ for all i. Verify that a map of complexes $\phi\colon A^\bullet \longrightarrow B^\bullet$ induces a map $\phi\colon H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ for all i by the rule $\phi(\bar{a}) = \overline{\phi(a)}$ for all $a \in \ker(\phi^i)$ (the bar denotes cosets).

Let $A^\bullet, B^\bullet, C^\bullet$ be three complexes in the Abelian category $\mathcal{C}$ and assume that there are morphisms $A^\bullet \xrightarrow{\phi} B^\bullet \xrightarrow{\psi} C^\bullet$ such that the sequences

$$0 \longrightarrow A^i \xrightarrow{\phi} B^i \xrightarrow{\psi} C^i \longrightarrow 0 \qquad (1)$$

are exact for all i. Then $0 \longrightarrow A^\bullet \xrightarrow{\phi} B^\bullet \xrightarrow{\psi} C^\bullet \longrightarrow 0$ is a *short exact sequence of complexes*. Moreover, in that case there is a *long exact sequence of cohomology*,

$$\ldots \longrightarrow H^i(A^\bullet) \xrightarrow{\phi} H^i(B^\bullet) \xrightarrow{\psi} H^i(C^\bullet) \xrightarrow{\delta} H^{i+1}(A^\bullet) \longrightarrow \cdots$$

that comes about as follows. Consider a chunk of the short exact sequence of complexes:

$$\begin{array}{ccccc} A^i & \xrightarrow{\phi^i} & B^i & \xrightarrow{\psi^i} & C^i \\ {\scriptstyle \alpha^i}\downarrow & & {\scriptstyle \beta^i}\downarrow & & {\scriptstyle \gamma^i}\downarrow \\ A^{i+1} & \xrightarrow{\phi^{i+1}} & B^{i+1} & \xrightarrow{\psi^{i+1}} & C^{i+1} \end{array}$$

The maps $\phi\colon H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ and $\psi\colon H^i(B^\bullet) \longrightarrow H^i(C^\bullet)$ are the ones discussed above. The *connecting morphisms* δ works as follows. Take $\bar{c} \in H^i(C^\bullet)$ and pick a representative $c \in \ker(\gamma^i)$. Then exactness of the sequence (1) shows that there is $b \in B^i$ with $\psi^i(b) = c$. As the diagram above is commutative, $\psi^{i+1}(\beta^i(b)) = \gamma^i(\psi^i(b)) = \gamma^i(c) = 0$. The exactness of the lower row implies that there is $a \in A^{i+1}$ with $\phi^{i+1}(a) = \beta^i(b)$. Note that $\alpha^{i+1}(a) = 0$ (why?!) and define $\delta(\bar{c}) = \overline{a}$. It is recommended that the reader prove that this map is independent of the various choices made.

1.2. Koszul complexes.

Example 1.1. Let R be a commutative ring, let $f \in R$, and take an R-module M. The *Koszul complex* of f on M is the complex

$$K^\bullet(f; M) = \left(M \xrightarrow{\cdot f} M \right).$$

Verify that $H^0(K^\bullet(f;M))$ is zero if and only if f is not a zerodivisor on M, and $H^i(K^\bullet(f;M)) = 0$ for all i if and only if f induces an isomorphism on M.

Exercise 1.2. *Let* $R = \mathbb{Z}$*,* $M = \mathbb{Z}/12\mathbb{Z}$*, and discuss* $K^\bullet(f;M)$ *for* $f = 0, 2, 3, 4, 5, 6$.

If $\boldsymbol{f} = f_1 \ldots, f_t \in R$ let $[t] = \{1, \ldots, t\}$. For $I \subseteq [t]$ and $j \in [t] \setminus I$ let $\operatorname{sgn}(I, j)$ be -1 raised to the number of elements in I that are preceded by j.

So $\operatorname{sgn}(\{1,3,4\},2) = (-1)^2$ as $1 < 2 < 3, 4$. There is a *Koszul complex* $K^\bullet(\boldsymbol{f}; M)$ whose k-th term is the direct sum

$$K^k(\boldsymbol{f}; M) = \bigoplus_{\substack{I \subseteq [t] \\ |I|=k}} R \cdot e_I$$

(where e_I is just a label), and where the map in the complex sends $1 \cdot e_I$ to

$$\sum_{j \in [t] \setminus I} \operatorname{sgn}(I, j) \cdot f_j \cdot e_{I \cup \{j\}}.$$

(Picture a t-dimensional hypercube, with vertices labeled by the powerset of $[t]$ (i.e., zero/one sequences of length t). The Koszul complex has modules to all vertices, maps to all edges, and the direct sum in each homological degree adds modules to vertices with equal label sum. For example, $K^\bullet(x, y; M)$ can be pictured as

$$\begin{array}{ccc} M & \xrightarrow{x} & M \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ M & \xrightarrow{-x} & M \end{array}$$

and where the upper right and lower left module form the term $K^1(x, y; M)$.)

Exercise 1.3. *Verify with $t = 2, 3$ that this definition makes $K^\bullet(\boldsymbol{f}; M)$ really a complex (i.e., $d \circ d = 0$).*

Compute the cohomology of $K^\bullet(xy, xz; R)$ where $R = \mathbb{C}[x, y, z]$.

1.3. **Algebraic geometry.** Recall the concept of an *ideal* in a ring.

Algebraic geometry deals with *varieties*, which come about as follows. Let R be a commutative ring (maybe think of a ring of polynomials over $\mathbb{C}$). Then the full collection of its prime ideals is called the *spectrum* of R. One can topologize it in the following funny way. Recall that a *topology* on a space X is a collection of open sets $\mathcal{T}$ such that all unions and all finite intersections of things in $\mathcal{T}$ give things in $\mathcal{T}$. Now let open sets in $\operatorname{Spec}(R)$ be the sets $U \subseteq \operatorname{Spec}(R)$ for which one can find a list of elements $\boldsymbol{f} = f_1, \ldots, f_t, \ldots$ in R such that for all prime ideals $\mathfrak{p}$ of R (i.e., for all elements of $\operatorname{Spec}(R)$) one has

$$[\mathfrak{p} \notin U] \Leftrightarrow [\boldsymbol{f} \subseteq \mathfrak{p}].$$

Verify that this in fact produces a topology, named after Oscar *Zariski*.

For a fixed collection $\boldsymbol{f} = \{f_1, \ldots, f_t, \ldots\}$, the set of primes $\mathfrak{p}$ with $\boldsymbol{f} \subseteq \mathfrak{p}$ is the *variety* of $\boldsymbol{f}$, denoted $\operatorname{Var}(\boldsymbol{f})$. Verify that if I is the ideal generated by $\boldsymbol{f}$ then $\operatorname{Var}(\boldsymbol{f}) = \operatorname{Var}(I)$.

If R is *Noetherian*, all increasing chains of ideals stabilize (have a maximal element). In particular, all sequences of ideals

$$Rf_1 \subseteq R(f_1, f_2) \subseteq R(f_1, f_2, f_3) \subseteq \cdots$$

stabilize. Hence in Notherian rings, one only has to deal with finite sets of generators $f_1, \ldots, f_t$. All rings we consider from now on will be Noetherian unless expressly indicated otherwise.

The natural association of a variety to a given ideal prompts the question whether this association is bijective. This is not so. Verify that x and x^2 determine the same variety but different ideals. More generally, show that if $I = R(f_1, \ldots, f_t)$

and $J = R(g_1, \dots, g_s)$ then $\mathrm{Var}(I) = \mathrm{Var}(J)$ if and only if there is $N \in \mathbb{N}$ such that $f_i^N \in J$ and $g_j^N \in I$ for all i, j. (Remember, we are in a Noetherian ring...)

If $R = \mathbb{K}[x_1, \dots, x_n]$ is a polynomial ring over a field $\mathbb{K}$, the elements of R represent certain functions on $\mathbb{K}^n$. (Warning: obviously not all functions can be represented this way–think of a way of showing that $\sin(x)$ is not represented by any polynomial! More warning: different polynomials may seem to be the same function. For example, compare the effect of x and x^p if $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$). The points of $\mathbb{K}^n$ correspond to certain *maximal ideals* of R ($\mathfrak{m}$ is maximal if there is only one ideal strictly larger than $\mathfrak{m}$, namely all of R), namely those of the form $R(x_1 - \alpha_1, \dots, x_n - \alpha_n)$ where α_i are the coordinates of a chosen point.

Note that R may contain other maximal ideals than those special ones. For example, the multiples of $x^2 + 1$ in $\mathbb{R}[x]$ form a maximal ideal $\mathfrak{m}$. Prove this in 2 steps. First show that if you have two ideals $I \subseteq J \neq R$ in a commutative ring then J/I is an ideal of R/I that is not equal to R/I. Then show that $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{C}$ by identifying $R/\mathfrak{m} \ni \bar{1} \leftrightarrow 1 \in \mathbb{C}$ and $R/\mathfrak{m} \ni \bar{x} \leftrightarrow i \in \mathbb{C}$, and conclude that in $R/\mathfrak{m}$ there is only one ideal that is not the whole ring.

The point of *Hilbert's Nullstellensatz* is that this aberration cannot happen in polynomial rings over $\mathbb{C}$. In other words, any maximal ideal in $\mathbb{C}[x_1, \dots, x_n]$ is of the form $(x_1 - \alpha_1, \dots, x_n - \alpha_n)$. This can be rephrased in many ways, for example by saying that any element of $\mathbb{C}[x_1, \dots, x_n]$ is either a constant, or vanishes in at least one point of $\mathbb{C}^n$. (Note how this reformulation fails in $\mathbb{R}$.) In fact, the Nullstellensatz does not require the underlying field $\mathbb{K}$ to be $\mathbb{C}$; it is sufficient if the field is *algebraically closed*: all single variable polynomials over $\mathbb{K}$ must split into linear factors).

Verify/review that we have the following dictionary between varieties and ideals:

radical ideal	$\leftrightarrow$	variety
radical of sum of ideals	$\leftrightarrow$	intersection of varieties
intersection of radical ideals	$\leftrightarrow$	union of varieties
prime ideal	$\leftrightarrow$	irreducible space
maximal ideals	$\leftrightarrow$	singletons

and, if $R = \mathbb{C}[x_1, \dots, x_n]$,

$$\text{maximal } R\text{-ideals} \leftrightarrow \text{points in } \mathbb{C}^n.$$

2. The Weyl algebra

We shall assume that $R = \mathbb{K}[x_1, \dots, x_n]$ where $\mathbb{K}$ is a field. The ring of $\mathbb{K}$-linear differential operators $D(R, \mathbb{K})$ of the commutative $\mathbb{K}$-algebra R is defined inductively: one sets $D_0(R, \mathbb{K}) = R$, and for $i > 0$ defines

$$D_i(R, \mathbb{K}) = \{P \in \mathrm{Hom}_{\mathbb{K}}(R, R) : Pr - rP \in D_{i-1}(R, \mathbb{K}) \forall r \in R\}.$$

Here, $r \in R$ is interpreted as the endomorphism of R that multiplies by r.

If the characteristic of $\mathbb{K}$ is zero, then $D = D(R, \mathbb{K})$ has the following simple description: D is the quotient of the free algebra

$$\mathbb{K}\langle x_1, \partial_1, \dots, x_n, \partial_n \rangle,$$

(consisting of $\mathbb{K}$-linear combinations of words in the letters $x_1, \ldots, \partial_n$ with composition as product) modulo the two-sided ideal generated by the relations

$$\begin{aligned} x_i x_j &= x_j x_i \quad \forall\, 1 \le i, j \le n, \\ \partial_i \partial_j &= \partial_j \partial_i \quad \forall\, 1 \le i, j \le n, \\ x_i \partial_j &= \partial_j x_i \quad \forall\, 1 \le i \ne j \le n, \\ \text{and } x_i \partial_i + 1 &= \partial_i x_i \quad \forall\, 1 \le i \le n. \end{aligned}$$

The last relation is nothing but the *product* (or *Leibniz*) *rule*, $xf' + f = (xf)'$.

In order to keep the product $\partial_i x_i \in D_n$ and the application of $\partial_i \in D_n$ to $x_i \in R_n$ apart, we shall write $\partial_i \bullet (g)$ to mean the result of the action of ∂_i on $g \in R_n$. So for example, $\partial_i x_i = x_i \partial_i + 1 \in D_n$ but $\partial_i \bullet x_i = \frac{\partial(x_i)}{\partial x_i} = 1 \in R_n$. The action of D_n on R_n takes precedence over the multiplication in R_n (and is of course compatible with the multiplication in D_n), so for example $\partial_2 \bullet (x_1) x_2 = 0 \cdot x_2 = 0 \in R_n$.

Remark 2.1. If $\mathbb{K}$ has positive characteristic, the corresponding Weyl algebra consists of course still of differential operators on R, but it does not fill out the entire ring $D(R, \mathbb{K})$. Verify that, in a single variable, the assignment

$$R \ni x^a \mapsto \begin{cases} \underbrace{\left(\dfrac{a(a-1)\cdots(a-p+1)}{p!}\right)}_{\in \mathbb{Z}} x^{a-p} & \text{if } a \ge p \\ 0 & \text{if } a < p. \end{cases}$$

is a differential operator that is not contained in $R\langle \partial_x \rangle$. (Often, this operator is mnemonically written as $\partial^p / p! \partial x^p$). It is suggested to try $p = 2$ first.

In fact, in positive characteristic $D(R, \mathbb{K})$ is not even Noetherian.

From now on assume that $\mathbb{K} \supseteq \mathbb{Q}$ and put $D = D(R, \mathbb{K})$. There is a class of modules, the *holonomic* ones, that are special. They are finitely generated modules with the extra condition that they have minimal dimension in the following sense. Give each x_i and each ∂_i the same weight $w = 1$ and introduce a filtration on D by

$$F^i(D) = \{P \in D, P \text{ is a sum of monomials of weight at most } i\}.$$

The *associated graded ring* $\operatorname{gr}(D)$ is by definition

$$\operatorname{gr}(D) = \bigoplus_{i \in \mathbb{N}} F^i(D)/F^{i-1}(D).$$

Verify that $\operatorname{gr}(D) \cong \mathbb{K}[\boldsymbol{x}, \boldsymbol{\partial}]$ is a polynomial ring.

By construction, the weight ω is now a degree function on $\operatorname{gr}(D)$. (The difference between weight and degree is that d is a degree function on a ring S if $S = \bigoplus_{d \in \mathbb{Z}} S_d$ and this decomposition respects multiplication and addition: $S_i S_j \subseteq S_{i+j}$ and $S_i + S_i \subseteq S_i$, while in a weight situation all you get is a filtration with $F_i(S) F_j(S) \subseteq F_{i+j}(S)$ but the direct sum decomposition is replaced by $S = \bigcup_{d \in \mathbb{Z}} S_d$. The weight $\omega(x_i) = \omega(\partial_i) = 1$ for all i is not a degree on D because x_i and ∂_i would be in the degree-1-piece while the element $\partial_i x_i - x_i \partial_i$ would have to be (homogenous and) of degree 2, but in actuality is of degree zero as $\partial_i x_i - x_i \partial_i = 1$.) This means that we can define a *Hilbert series*

$$H(F, M, t) = \sum_{i \in \mathbb{Z}} \dim(\operatorname{gr}^i(M)) t^i,$$

where the filtration we used to get a graded object is listed in the arguments. Asymptotically, $\dim(\mathrm{gr}^i(M))$ is a polynomial expression in i, called the *Hilbert-polynomial* $P(F, M, i)$. The degree of this polynomial is at most $2n-1$ since that is what you get for $M = D$. (Verify this!) The *F-dimension* $\dim_F(M)$ *of* M is the degree of $P(F, M, i)$ plus 1.

A famous theorem (by I.N. Bernstein)[1] says that $\dim_F(M)$ cannot be smaller than n. Note that such theorem cannot hold over commutative rings since for an arbitrary maximal ideal $\mathfrak{m}$ in a commutative ring R the quotient $R/\mathfrak{m}$ is of dimension zero. The news is that D has no maximal ideals in the old sense.

Holonomic modules are those of this minimal F-dimension n. Any such module has finite length as D-module, is generated by a single element, and both Artinian and Noetherian. (Make sense of these statements and verify them explicitly in the case $M = R$. In particular, show that for all $f \in R$ one can produce 1 as element of the left submodule $D \bullet f \subseteq R$.)

Alternatively, if $M = D/I$ is a left D-module then the ideal $\mathrm{gr}^F(I)$ is an ideal in $\mathrm{gr}^F(D)$, a $2n$-dimensional commutative ring of polynomials. Hence $\mathrm{Var}(\mathrm{gr}^F(I))$ is an algebraic variety in $\mathbb{K}^{2n}$ is the *characteristic variety* of M; its dimension agrees with $\dim_F(M)$ as defined above. As an exercise, compute $\dim_F(D)$, $\dim_F(R)$, $\dim_F(D/(x\partial + 1))$ in the case $n = 1$.

Somewhat unbelievably, if M is holonomic then so is its localization $M[1/f] := R[1/f] \otimes_R M$. In particular, such a localization $R[1/f]$ is finitely generated over D. Contemplate this if $M = R = \mathbb{C}[x, y, z, w]$ in general, and specifically if $f = x$ and then if $f = x^2 + y^2 + z^2 + w^2$. (That means, find a D-generator for $R[1/f]$. You may need help on the second part; see [Bjö79, Bjö93])

References

[Bjö79] J.-E. Björk, *Rings of differential operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam, 1979. MR **82g:**32013

[Bjö93] Jan-Erik Björk, *Analytic $\mathcal{D}$-modules and applications*, Mathematics and its Applications, vol. 247, Kluwer Academic Publishers Group, Dordrecht, 1993. MR **95f:**32014

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[1]because of transliteration issues, J. Bernstein, I.N. Bernstein, and И. Бернштейн are all the same person.