Geometric Objects on Fibered Manifolds and Integrable Systems

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Prof. WILHELM PLESKEN's 60th birthday,
RWTH-Aachen, May 2010

The motivation

Example:

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- And, as such, even integrable!

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The Lie algebra of vector fields $D := \Gamma(TM)$ acts on the spaces of sections $\mathcal{F} \in \{\Gamma(TM), \Gamma(T^*M), \Gamma(\bigwedge^2 TM), \ldots\}$ via the Lie derivative: $\mathcal{L} : D \to \mathrm{End}(\mathcal{F})$.

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super Lie bracket $[\Theta, \Xi]$ for $\Theta \in \Gamma(\bigwedge^k TM), \Xi \in \Gamma(\bigwedge^l TM)$	(funct.) super Lie bracket $[\Theta, \Xi]$ for $\Theta \in \bigwedge^k \mathcal{V}^1, \Xi \in \bigwedge^l \mathcal{V}^1$

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- Implemented it in the Maple package `jets` (joint work with GEHRT HARTJEN).

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The BOUSSINESQ as an ODE

Now consider the BOUSSINESQ functional vector field

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A Theorem and a Conjecture

Theorem

Let ξ be the BOUSSINESQ *functional vector field. The space of ξ-invariant $(2,0)$-functional* ***tensors*** *of order ≤ 10 and jet order ≤ 6 is a* ***two*** *dimensional $\mathbb{R}$-linear space generated by the two functional* ***bivectors***

$$\Theta := \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

$$\Xi := \begin{pmatrix} D_x^3+2uD_x+u_x & 3vD_x+2v_x \\ 3vD_x+v_x & \frac{1}{3}D_x^5+\frac{5}{3}(uD_x^3+D_x^3\cdot u)-(u_{xx}D_x+D_x\cdot u_{xx})+\frac{16}{3}uD_x\cdot u \end{pmatrix}.$$

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A Theorem and a Conjecture

Theorem + Conjecture

Let ξ be the BOUSSINESQ *functional vector field. The space of ξ-invariant $(2,0)$-functional* ***tensors*** ~~*of order ≤ 10 and jet order ≤ 6*~~ *is a* ***two*** *dimensional $\mathbb{R}$-linear space generated by the two functional* ***bivectors***

$$\Theta := \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

$$\Xi := \begin{pmatrix} D_x^3+2uD_x+u_x & 3vD_x+2v_x \\ 3vD_x+v_x & \frac{1}{3}D_x^5+\frac{5}{3}(uD_x^3+D_x^3\cdot u)-(u_{xx}D_x+D_x\cdot u_{xx})+\frac{16}{3}uD_x\cdot u \end{pmatrix}.$$

Moreover, these two functional bivectors are ***compatible*** **POISSON** *bivectors:*

$$[\Theta,\Theta]=0, [\Xi,\Xi]=0, [\Theta,\Xi]=0.$$

Surprising phenomenon

The functional calculus admits no “rectification theorem”.

Surprising phenomenon

The functional calculus admits no “rectification theorem”. It is an extremely rigid calculus with close ties to combinatorial structures in algebraic geometry.