Functional Spaces.
A Direct Approach.

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Preface

First of all I want to thank God and my small family, especially my wife for her encouragement and patience. To Professor Plesken for his time, his interest and his advice. I would also like to thank Professor Ian Anderson for giving us access to his unpublished book. His and Professor Olver's books were of great influence on this work. Last but not least, I want to thank Gehrt Hartjen, with whom I developed the Maple package `jets`, which is the tool I used to perform all the computations appearing in [Olv] and this thesis.

Introduction

This work deals with the calculus of *functional* spaces, which generalizes the classical tensor calculus on manifolds. Especially the spaces of functional s-forms $\mathcal{F}^s$, functional s-vectors $\mathcal{V}^s$, and also mixed functional tensor spaces like the space of functional $(1,1)$-tensors $\mathcal{F}^1 \otimes \mathcal{V}^1$ are introduced.

Functional spaces are of major importance in the formal theory of PDEs, the simplest and well known examples being $\mathcal{F}^0$, $\mathcal{F}^1$ and $\mathcal{V}^1$. The space $\mathcal{F}^0$ is nothing but the space of functionals, i.e. integrals of Lagrangians. The space $\mathcal{F}^1$, also called the space of source forms, contains as a subspace the wide class of Euler-Lagrange equations. The "dual" space $\mathcal{V}^1$, also called the space of *evolutionary* or *characteristic* vector fields, was already known to Lie. But also higher functional spaces appeared recently in a natural manner. The classical Helmholtz conditions of the calculus of variations have been recognised to be a functional 2-form, i.e. an element of $\mathcal{F}^2$. The Hamiltonian structures of completely integrable evolution equations are functional 2-vectors, i.e. elements of $\mathcal{V}^2$, and the Lenard type recursion operators are functional $(1,1)$-tensors, i.e. elements of $\mathcal{F}^1 \otimes \mathcal{V}^1$. The most popular examples of such equations are the Korteweg-de Vries, Boussinesq and nonlinear Schrödinger equation.

The formal setting for all these objects is the *jet calculus*. It has already been used by Lie in an informal manner to have coordinates for partial derivatives of all orders. For instance a vector field on a fibred manifold with components depending not only on the original coordinates, but also on their formal derivatives up to a certain order, is called *generalized* vector field. Details on fibred manifolds and associated jet spaces will be introduced later. The jet language will be freely used throughout this thesis. Good references are [And] and [Pom].

The first aim is to define functional spaces using a minimum of unfamiliar language. We define higher functional forms as totally skew-symmetric matrix differential operators, avoiding the introduction of differential forms[1], appearing in [And] and [Olv], to define them. We define functional multi-vectors and mixed tensors in a similar fashion. This *practically oriented* rather than intrinsic definition yields a simple data structure for computer implementations and a concise

[1] For the connection between the operator approach adopted here (for *all* functional spaces) and the differential form approach for functional 1- and 2-forms see [Olv], Propositions 5.87 & 5.88.

notation for theoretical investigations. As opposed to the differential form approach it enables us to work with forms, multi-vectors and tensors on an equal footing. Furthermore, even with the aid of the LEIBNIZ product rule *alone*, we are able to recursively determine the action of generalized infinitesimal transformations, i.e. the Lie derivative of generalized vector fields on these spaces, adding to them the inner life stolen by our nonintrinsic definition. This is explicitly done for all functional tensor spaces $\bigotimes^r \mathcal{F}^1 \otimes \bigotimes^s \mathcal{V}^1$ including the sub-cases $\mathcal{F}^s$ and $\mathcal{V}^s$. Algebraically speaking, the basic principle of this approach, namely the LEIBNIZ product rule, recursively turns these spaces into modules for the Lie algebra of generalized vector fields, or equivalently for the Lie algebra of evolutionary vector fields $\mathcal{V}^1$, starting with $\mathcal{F}^0$ and proceeding to higher order functional spaces.

The second and more ambitious aim is to construct in a straightforward manner the so called EULER complex

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\delta} \mathcal{F}^3 \xrightarrow{\delta} \dots$$

This is an infinite, locally exact sequence of spaces of functional forms, generalizing the DE RAHM complex from differential geometry.

In the new approach taken here one defines the EULER complex by assuming the validity of the CARTAN formula

$$\mathcal{L}_X = \delta\, \iota_X + \iota_X \delta$$

at each step, where $\mathcal{L}_X : \mathcal{F}^s \longrightarrow \mathcal{F}^s$ is the Lie derivative along the generalized vector field X and $\iota_X : \mathcal{F}^{s+1} \longrightarrow \mathcal{F}^s$ is the interior product with respect to X. This suffices to recursively define the morphisms δ at each stage. At the same time it automatically enforces[2] the *local exactness* of the constructed sequence. The first three steps of this recursion are explicitly carried out. The first operator $\delta = \mathsf{E} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1$ coincides with the EULER operator of the classical variational calculus. The EULER complex owes its name to this fact. The second one $\delta = \mathsf{H} : \mathcal{F}^1 \longrightarrow \mathcal{F}^2$ reproduces the HELMHOLTZ conditions of the variational calculus. It is called the HELMHOLTZ operator following [And]. The simple description of the space $\mathcal{F}^3$ and the recursive approach yields a description of the operator $\delta = \mathsf{T} : \mathcal{F}^2 \longrightarrow \mathcal{F}^3$. I call it TAKENS operator.

The EULER sequence has an exact formal analogy with the DE RAHM sequence, except that the former does not terminate and that there is no analogue for the wedge product. Nevertheless, if we allow the number of *independent* variables to be zero, then both languages coincide. A functional is a function, a functional form is a form, and so on. Further E becomes the gradient, H becomes the curl, and so forth. The EULER complex is then precisely the DE RAHM complex.

[2] The integrated form of a Cartan formula is a homotopy formula expressing the local exactness.

The Euler complex appears naturally as the rightmost column of the *variational bicomplex*, which is avoided by this direct approach, but nevertheless a central object in what I. M. Gel'fand called "formal differential geometry" in his 1970 address to the International Congress in Nice. It is a locally exact double complex.

$$
\begin{array}{ccccccccccccccccc}
 & & & & \uparrow\delta & & & & & & & & \uparrow\delta & & \uparrow\delta & & \\
 & & 0 & \longrightarrow & \Omega^{0,3} & & & & \cdots & & & & \Omega^{p,3} & \xrightarrow{\int_M} & \mathcal{F}^3 & \longrightarrow & 0 \\
 & & & & \uparrow\delta & & & & & & & & \uparrow\delta & & \uparrow\delta & & \\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{D} & \Omega^{1,2} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \Omega^{p-1,2} & \xrightarrow{D} & \Omega^{p,2} & \xrightarrow{\int_M} & \mathcal{F}^2 & \longrightarrow & 0 \\
 & & & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{D} & \Omega^{1,1} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \Omega^{p-1,1} & \xrightarrow{D} & \Omega^{p,1} & \xrightarrow{\int_M} & \mathcal{F}^1 & \longrightarrow & 0 \\
 & & & & \uparrow\delta & & \uparrow\delta & & & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{D} & \Omega^{1,0} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \Omega^{p-1,0} & \xrightarrow{D} & \Omega^{p,0} & \xrightarrow{\int_M} & \mathcal{F}^0 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^0_M & \xrightarrow{d} & \Omega^1_M & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{p-1}_M & \xrightarrow{d} & \Omega^p_M & \longrightarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & \cdots & & 0 & & 0 & & & &
\end{array}
$$

Starting with a fibred manifold $\pi : \mathcal{E} \to M$ over a p-dimensional base manifold M one defines jets, contact forms and horizontal forms, and uses them to introduce the spaces of graded differential forms $\Omega^{r,s} = \Omega^{r,s}(\pi)$. In the above diagram D denotes the *total* derivative and δ the *vertical* derivative, naturally extended to differential forms. Their sum $d = D + \delta$ is the *exterior* derivative on the infinite jet space $J^\infty(\pi)$. The bottom row is the classical de Rahm complex of the base manifold M.

The terminology "variational bicomplex" is motivated by the above mentioned facts that $\delta : \mathcal{F}^0 \to \mathcal{F}^1$ coincides with the Euler operator and $\delta : \mathcal{F}^1 \to \mathcal{F}^2$ reproduces the Helmholtz conditions of the calculus of variations. Nevertheless there is a wide range of applications of the variational bicomplex that go far beyond variational problems. For details see [And].

The variational bicomplex provides a way to intrinsically define $\mathcal{F}^s$ as the quotient space $\Omega^{p,s}/D(\Omega^{p-1,s})$. Because the last D coincides with the divergence, one could naturally identify the quotient map $\Omega^{p,s} \to \mathcal{F}^s$ with the integral over M. Due to this fact, one calls these spaces "functional" spaces. This approach is used in [Olv]. An equivalent approch relying on a projection operator $I : \Omega^{p,s} \to \Omega^{p,s}$ called the "interior Euler operator" and defining $\mathcal{F}^s := I(\Omega^{p,s})$, i.e. as a subspace of $\Omega^{p,s}$, can be found in [And].

In the dual context we define the Nijenhuis-Schouten bracket for functional multi-vectors, generalizing the classical one. We use it to define the notion of a Hamiltonian system of evolution equations. As we succeed to find a useful normal form for functional 3-vectors, we can check the Hamiltonian condition in a direct manner. We further use the simple idea, that the Hamiltonian structure of a Hamilton equation is, as a functional 2-vector, invariant under the flow of the equation, to determine all Hamiltonian structures, up to a certain order, of some known nonlinear completely integrable differential equations. Because we cannot parametrize the nonlinear space of Hamiltonian 2-vectors, we instead compute the space of functional $(2, 0)$-tensors, invariant under the flow, and then determine the Hamiltonian 2-vectors among them. Using these results, recursion operators are easily constructed. This is the third and last aim of this thesis.

Contents

Chapter 1

Functional Spaces

1.1 Basic Definitions

Let $E \to M$ be a fibred manifold in p independent variables $(x^i) = (x^1, \ldots, x^p)$ and q dependent variables $(u^\alpha) = (u^1, \ldots, u^q)$ (cf. [Pom]). Let $\pi_M^\infty : J_\infty(E) \to M$ denote the infinite *jet bundle* having the *jet variables* (x^i, u_J^α) as coordinates, where $J = (J_1, \ldots, J_p)$ is an arbitrary multi-index[1]. So in the case of two independent variables (x, y), the jet variable u is addressed by $u_{(0,0)}$, u_y by $u_{(0,1)}$, and u_{xxy} by $u_{(2,1)}$.

Note that higher jet coordinates transform like derivatives, i.e. they transform according to the *chain rule*, nevertheless they are *not* derivatives:

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial u_x}{\partial x} = \frac{\partial u_{xx}}{\partial x} = \ldots = \frac{\partial u_x}{\partial u} = \ldots = 0.$$

Since global aspects do not play any role in this work, it suffices to express all objects locally, i.e. in terms of a fixed coordinate system, provided one knows how the objects transform under one parameter subgroups of transformations (or more generally under coordinate changes of E). Since a one parameter subgroup is determined by its infinitesimal transformation in the sense of Lie, it suffices to consider infinitesimal transformations (i.e. *Lie derivatives*) of the objects, which is the philosophy adopted in this work. They have the advantage of being linear operations compared to the highly nonlinear coordinate change formulas, inheriting their nonlinearity from the repeated use of the chain rule.

Following [Olv], let $\mathcal{A}$ denote[2] the space of *differential expressions* over E, i.e. smooth real-valued functions of finitely many arbitrary jet variables. A jet expression $f = f(x^i, u_J^\alpha)$ is abbreviated by $f = f[u]$. For example

$$f[u] = \cos(x)\sqrt{1 + u_y^2} + ye^{u^2 u_{xxy}}.$$

[1] [Olv] uses a different notion of multi-index: $u_{(2)}$ stands for u_y and $u_{(1,1,2)}$ for u_{xxy}.

[2] Depending on the context, $\mathcal{A}$ will stand for $\Omega^{0,0}$ or $\Omega^{p,0}$ in the language of the variational bicomplex. $\Omega^{p,0}$ is the space of horizontal p-forms.

Throughout this work[3]

$$D_i = D_{x^i} := \frac{\partial}{\partial x^i} + u^\alpha_{J+1_i}\frac{\partial}{\partial u^\alpha_J} \tag{1.1}$$

denotes the *total derivative* with respect to x^i, $i = 1, \dots, p$, where $J + 1_i := (J_1, \dots, J_i + 1, \dots, J_p)$. Further define

$$D_J := (D_1)^{J_1} \cdots (D_p)^{J_p}. \tag{1.2}$$

By this one gets for example

$$u_{xxy} = D_x u_{xy} = D_y u_{xx} = D_{xx} u_y = D_{xxy} u = \frac{D_{xx}(x u_y) - 2u_{xy}}{x}.$$

A *generalized vector field*[4] $\mathbf{v}$ is a vector field on E which may depend on higher jet variables

$$\mathbf{v} = \xi^i[u]\frac{\partial}{\partial x^i} + \eta^\alpha[u]\frac{\partial}{\partial u^\alpha}.$$

The *characteristic* of a vector field $\mathbf{v}$ is a column $Q \in \mathcal{A}^{q\times 1}$ defined by

$$Q^\alpha := \eta^\alpha - u^\alpha_i \xi^i. \tag{1.3}$$

And conversely to each characteristic $Q \in \mathcal{A}^{q\times 1}$ one associates a vector field

$$\mathbf{v}_Q := Q^\alpha \frac{\partial}{\partial u^\alpha}, \tag{1.4}$$

called the *evolutionary* or *characteristic vector field* with characteristic Q. If Q is the characteristic of a vector field $\mathbf{v}$, then one calls $\mathbf{v}_Q$ the characteristic vector field associated to $\mathbf{v}$ and denotes it occasionally by $\mathbf{v}_{\mathrm{ev}}$.
Let for example (t, x) be the independent, and (u, v) the dependent variables. If

$$\mathbf{v} = \partial_t + v_x\partial_u + (\frac{1}{3}u_{xxx} + \frac{8}{3}uu_x)\partial_v,$$

then

$$Q = \begin{pmatrix} Q^u \\ Q^v \end{pmatrix} = \begin{pmatrix} v_x - u_t \\ \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x - v_t \end{pmatrix},$$

and

$$\mathbf{v}_Q = (v_x - u_t)\partial_u + (\frac{1}{3}u_{xxx} + \frac{8}{3}uu_x - v_t)\partial_v.$$

For a characteristic vector field $\mathbf{v}_Q$ the (infinite) *prolongation* is defined by

$$\mathsf{pr}\,\mathbf{v}_Q := D_J Q^\alpha \frac{\partial}{\partial u^\alpha_J}. \tag{1.5}$$

[3] The sum is always to be taken over equal upper and lower indices (summation convention).
[4] The word “generalized” is often skipped in the text.

And for a generalized vector field $\mathbf{v} = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}$ with characteristic Q one defines

$$\mathsf{pr}\,\mathbf{v} := \mathsf{pr}\,\mathbf{v}_Q + \xi^i D_i. \tag{1.6}$$

The prolongation formula arises as follows: A vector field on E generates a local flow on E, which determines by chain rule a so-called *prolonged* local flow on $J_\infty(E)$, having the prolonged vector field as its infinitesimal transformation. Thus the prolongation formula is nothing but the *infinitesimal chain rule*.
When applying generalized vector fields to differential expressions, one must first prolong the vector field before applying it. Since the prolongation $\mathsf{pr}\,\mathbf{v}$ is uniquely determined by $\mathbf{v}$ and conversely $\mathbf{v}$ is a part of $\mathsf{pr}\,\mathbf{v}$, we shall identify them. Note, the prolongation operator pr is $\mathbb{R}$-linear and *not* $\mathcal{A}$-linear.

For the above example one verifies

$$\begin{aligned}\mathsf{pr}^{(1)}\mathbf{v}_Q = {} & (v_x - u_t)\partial_u + (\frac{1}{3}u_{xxx} + \frac{8}{3}uu_x - v_t)\partial_v \\ & +(v_{tx} - u_{tt})\partial_{u_t} + (\frac{1}{3}u_{txxx} + \frac{8}{3}u_tu_x + \frac{8}{3}uu_{tx} - v_{tt})\partial_{v_t} \\ & +(v_{xx} - u_{tx})\partial_{u_x} + (\frac{1}{3}u_{xxxx} + \frac{8}{3}u_x^2 + \frac{8}{3}uu_{xx} - v_{tx})\partial_{v_x}\end{aligned}$$

and

$$\begin{aligned}\mathsf{pr}^{(1)}\mathbf{v} = {} & \partial_t + v_x\partial_u + (\frac{1}{3}u_{xxx} + \frac{8}{3}uu_x)\partial_v \\ & +v_{tx}\partial_{u_t} + (\frac{1}{3}u_{txxx} + \frac{8}{3}u_tu_x + \frac{8}{3}uu_{tx})\partial_{v_t} \\ & +v_{xx}\partial_{u_x} + (\frac{1}{3}u_{xxxx} + \frac{8}{3}u_x^2 + \frac{8}{3}uu_{xx})\partial_{v_x},\end{aligned}$$

where $\mathsf{pr}^{(k)}\mathbf{v}$ means the k-th prolongation of $\mathbf{v}$, i.e. the prolongation up to the partial derivatives w.r.t. jets of order at most k.

1.2 Functional Spaces

1.2.1 Definition (Total differential operator)

A *total*[5] *(matrix) differential operator* is an operator of the form $\mathcal{D} = (P^J_{\alpha\beta} D_J)$ ($\alpha = 1, \ldots, r$, $\beta = 1, \ldots, s$, finite sum over J)

$$\mathcal{D} : \left\{ \begin{array}{ccc} \mathcal{A}^r & \to & \mathcal{A}^s \\ (T_\alpha)_\alpha & \mapsto & (\sum_{\alpha=1}^r P^J_{\alpha\beta} D_J T_\alpha)_\beta \end{array} \right.$$

where $P^J_{\alpha\beta} = P^J_{\alpha\beta}[u] \in \mathcal{A}$. The *order* of a differential operator is the largest number m with at least one $P^J_{\alpha\beta} \neq 0$ for $|J| = m$. Its *jet order* is the highest order of a jet variable appearing in the coefficients $P^J_{\alpha\beta}$ of $\mathcal{D}$.

[5]The word "total" is often dropped in the sequel.

1.2.2 Remark
$\mathcal{A}^k$ stands for the space of *k-tuples*. At this stage, the position of the indices (α and β being upper or lower) does not play a role. Once we identify $\mathcal{A}^k$ with either the column space $\mathcal{A}^{k\times 1}$ or the row space $\mathcal{A}^{1\times k}$, then the index position matters (see below). To apply a matrix differential operator to a row or a column T, one identifies T back with a column tuple, only to be able to interpret $\mathcal{D}T$ as a matrix applied to a column.

1.2.3 Example
The following[6] matrix operator $\mathcal{E}$

$$\mathcal{E} = \begin{pmatrix} D_x^3 + (uD_x + D_x \cdot u) & 3vD_x + 2v_x \\ 3vD_x + v_x & \frac{1}{3}D_x^5 + \frac{5}{3}(uD_x^3 + D_x^3 \cdot u) - (u_{xx}D_x + D_x \cdot u_{xx}) + \frac{16}{3}uD_x \cdot u \end{pmatrix}.$$

has order 5 and jet order 3.

For details and motivation of the below definitions see [Olv], Chapter 5, Section 4.

1.2.4 Definition (Current)
A p-tuple $A = (A^1, \dots, A^p) \in \mathcal{A}^p$ of differential expressions is called a *current* or a *horizontal* $(p-1)$*-form*[7], where p is the number of *independent* variables $(x^1, \dots, x^p)$.

1.2.5 Definition (Divergence)
For a current $A = (A^1, \dots, A^p)$ the operator $\mathsf{Div} : \mathcal{A}^p \to \mathcal{A}$ defined by

$$\mathsf{Div}A := D_i A^i \tag{1.7}$$

is called the *divergence* operator.

1.2.6 Lemma (Integration by parts formula)
For a matrix differential operator $\mathcal{D} = (\mathcal{D}_{\alpha\beta}) : \mathcal{A}^r \to \mathcal{A}^s$ and for $S \in \mathcal{A}^s$, there exists a unique *$R \in \mathcal{A}^r$, such that for each $Q \in \mathcal{A}^r$, there exists a current $A \in \mathcal{A}^p$, such that the following* integration by parts formula *holds*[8]

$$S \cdot \mathcal{D}Q = R \cdot Q + \mathsf{Div}A.$$

More explicitly, for $\mathcal{D}_{\alpha\beta} = P^J_{\alpha\beta}D_J$ and $S = (S_\beta)$ the unique $R = (R_\alpha)$ is given by

$$R_\alpha = \sum_\beta (-D)_J (P^J_{\alpha\beta} S_\beta),$$

where $(-D)_J := (-1)^{|J|} D_J$.

[6] $D_x \cdot u := D_x(u\,\cdot)$
[7] $\mathcal{A}^p$ stands for $\Omega^{p-1,0}$ in the language of the variational bicomplex.
[8] $R \cdot Q := \sum_\alpha R_\alpha Q_\alpha$.

Proof. Cf. [Olv], Section 5.3. □

1.2.7 Definition (Adjoint operator)
Using the notation of the above lemma, the map $S \mapsto R$ defines a total differential operator $\mathcal{D}^* = (\mathcal{D}^*_{\beta\alpha}) : \mathcal{A}^s \to \mathcal{A}^r$ called the formal *adjoint operator* of $\mathcal{D}$:

$$\mathcal{D}^*_{\beta\alpha} := (-D)_J(P^J_{\alpha\beta}\ \cdot).$$

Note, the adjoint of a matrix differential operator is the transposed of the matrix where the adjoint operator is applied componentwise.

For example, if

$$\mathcal{D} = D_{xy} + u_y D_x + u_{xy},$$

then its adjoint is

$$\mathcal{D}^* = (-1)^2 D_y D_x + (-D_x)u_y + u_{xy} = D_{xy} - u_y D_x.$$

Every operator coincides with its double adjoint:

$$\mathcal{D}^{**} = D_{xy} + D_x u_y = D_{xy} + u_y D_x + u_{xy}$$

1.2.8 Corollary (Integration by parts formula)
The integration by parts formula takes the form[9]

$$S \cdot \mathcal{D}Q = \mathcal{D}^* S \cdot Q + \mathsf{Div} A. \tag{1.8}$$

1.2.9 Definition (Self- and skew-adjointness)
An operator $\mathcal{D}$ is *self-adjoint* (resp. *skew-adjoint*), if $\mathcal{D}^* = \mathcal{D}$ (resp. $\mathcal{D}^* = -\mathcal{D}$).

1.2.10 Example
The operator $\mathcal{E}$ of Example 1.2.3 (see also Example 4.3.14) is written in such a manner, that its skew-adjointness becomes obvious.

The composition of two total differential operators is again a total differential operator.

1.2.11 Lemma
Let $\mathcal{D} = P^J D_J : \mathcal{A} \to \mathcal{A}$ and $\mathcal{E} = M^I D_I : \mathcal{A} \to \mathcal{A}$ be two differential operators. Then

$$\mathcal{D}\mathcal{E} = \sum_K \binom{I}{K} P^J D_K(M^I) D_{I+J-K}, \tag{1.9}$$

where $\binom{I}{K} := \frac{I!}{K!(I-K)!}$ in multi-index notation. The composition of two matrix differential operators is the usual matrix product, where the multiplication of entries is replaced by the composition (1.9).

[9] Cf. [Olv], Formula (5.77).

PROOF. This is a direct consequence of the Leibniz rule for total derivatives D_{x^i} and the fact that they commute pairwise. □

1.2.12 Example
For $\mathcal{D} = D_{xxx} + uD_x + u_x$ and $\mathcal{E} = D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$ one verifies for the product operator

$$\mathcal{D}\mathcal{E} = D_x^6 + \frac{5}{3}uD_x^4 + \frac{10}{3}u_xD_x^3 + (3u_{xx} + \frac{2}{3}u^2)D_x^2 + (\frac{5}{3}u_{xxx} + \frac{5}{3}uu_x)D_x + (\frac{1}{3}u_{xxxx} + \frac{1}{3}uu_{xx} + \frac{1}{3}u_x^2).$$

And for the adjoint operator $\mathcal{D}^* = -D_{xxx} - uD_x$ one verifies similarly

$$\mathcal{E}\mathcal{D}^* = -D_x^6 - \frac{5}{3}uD_x^4 - \frac{10}{3}u_xD_x^3 - (3u_{xx} + \frac{2}{3}u^2)D_x^2 - (u_{xxx} + uu_x)D_x.$$

See also Example 1.3.6.

1.2.13 Corollary
The adjoint of the product of two composable differential operators $\mathcal{D}, \mathcal{E}$ satisfies

$$(\mathcal{D}\mathcal{E})^* = \mathcal{E}^*\mathcal{D}^*. \tag{1.10}$$

PROOF. General integration by parts formula (1.8). □

1.2.14 Definition (Functionals, Lagrangians)
The space $\mathcal{F}^0$ defined locally[10] by

$$\mathcal{F}^0 := \mathcal{A}/\mathsf{Div}(\mathcal{A}^p)$$

is called the space of *functionals*. In this context[11], elements of $\mathcal{A}$ are called *Lagrangians*. Lagrangians representing the same functional are called *equivalent*.

For example the two Lagrangians $\frac{1}{2}u_x^2$ and $-\frac{1}{2}uu_{xx}$ are one and the same functional; they differ by $D_x(\frac{1}{2}uu_x)$.

1.2.15 Definition (Functional 1-vectors)
By $\mathcal{V}^1$ we denote the space of evolutionary vector fields, or equivalently the space of *characteristics* over a jet bundle. This space can be identified with $\mathcal{A}^{q\times 1}$. Its elements are also called *functional vectors*, or more elaborately *functional* 1-*vectors*. For the components of a characteristic $Q = (Q^\alpha)$ one uses an upper index and calls the components *contravariant*.

[10] Two Lagrangians represent the same functional, if and only if their difference is locally a divergence.

[11] Here $\mathcal{A}$ stands for $\Omega^{p,0}$: $L \in \mathcal{A}$ stands for the horizontal p-form $Ldx^1 \wedge \ldots \wedge dx^p$.

By $\mathcal{D} : \mathcal{V}^1 \to \mathcal{A}^r$ we mean an $r \times q$-matrix[12] of differential operators $\mathcal{D} = (\mathcal{D}^l_\alpha)^{l=1,\dots,r}_{\alpha=1,\dots,q}$ acting on characteristics via

$$(\mathcal{D}Q)^l = \mathcal{D}^l_\alpha Q^\alpha, \quad l = 1, \dots, r.$$

In order to define the $\mathcal{F}^0$-dual of $\mathcal{V}^1$ we need the following special case of the integration by parts formula:

1.2.16 Corollary
For $Q \in \mathcal{V}^1$ and $\mathcal{D} : \mathcal{V}^1 \to \mathcal{A}$ there exists a current $A \in \mathcal{A}^p$ such that the following integration by parts *formula holds*

$$\mathcal{D}Q = \mathcal{D}^*(1) \cdot Q + \mathsf{Div}A, \tag{1.11}$$

where $\mathcal{D}^*(1) = (\mathcal{D}^*_1(1), \dots, \mathcal{D}^*_q(1)) \in \mathcal{A}^{1\times q}$. This decomposition is unique[13]*, in the sense that if there exists a $\Delta \in \mathcal{A}^{1\times q}$, such that for each $Q \in \mathcal{V}^1$, there exists a current $B \in \mathcal{A}^p$, such that $\mathcal{D}Q = \Delta \cdot Q + \mathsf{Div}B$, then $\Delta = \mathcal{D}^*(1)$.*

PROOF. This is a special case of the general integration by parts formula (1.8) and the uniqueness property of Lemma 1.2.6. □

For the example following 1.2.7, $\mathcal{D}^*(1) = 0$, thus $\mathcal{D}Q$ must be a divergence for all $Q \in \mathcal{V}^1$. Indeed $\mathcal{D}Q = \mathsf{Div}A$ with $A = (u_y Q, D_x Q)$, but also $\mathcal{D}Q = \mathsf{Div}B$ with $B = (u_y Q + D_y Q, 0)$.

1.2.17 Corollary
A differential operator $\mathcal{D} : \mathcal{V}^1 \to \mathcal{A}$ viewed as differential operator $\mathcal{D} : \mathcal{V}^1 \to \mathcal{F}^0$ can be identified with a unique element in $\mathcal{A}^{1\times q}$. Conversely any element $\Delta \in \mathcal{A}^{1\times q}$ defines via $Q \mapsto \Delta \cdot Q := \Delta_\alpha Q^\alpha$ a differential operator $\mathcal{V}^1 \to \mathcal{F}^0$.

1.2.18 Definition (Source forms, Functional 1-forms)
The space $\mathcal{F}^1 := \mathrm{Hom}(\mathcal{V}^1, \mathcal{F}^0) := \{\mathcal{D} : \mathcal{V}^1 \to \mathcal{F}^0 |\ \mathcal{D}$ total differential operator$\}$ is called the $\mathcal{F}^0$-dual space of $\mathcal{V}^1$. It can be identified with $\mathcal{A}^{1\times q}$. Its elements are called *source forms* or *functional forms*, or more elaborately *functional* 1*-forms*. For the components of a source form $\Delta = (\Delta_\alpha)$ one uses a lower index and calls the components *covariant*.

The integration by parts formula (1.11) states that the $\mathbb{R}$-bilinear map

$$\begin{array}{ccc} \mathcal{F}^1 \times \mathcal{V}^1 & \to & \mathcal{F}^0 \\ (\Delta, Q) & \mapsto & \Delta \cdot Q \end{array}$$

is a natural $\mathcal{F}^0$-valued pairing[14] of $\mathcal{V}^1$ and $\mathcal{F}^1$, thus $(\mathcal{V}^1, \mathcal{F}^1)$ is a $\mathcal{F}^0$-pair, and we write $(\mathcal{V}^1)^* \cong \mathcal{F}^1$ and $(\mathcal{F}^1)^* \cong \mathcal{V}^1$.

[12] The position of the index α for $\mathcal{D}$ and Q is important, whereas the position of l is yet irrelevant.
[13] Although the current A, depending on $\mathcal{D}$ and Q, is not unique.
[14] A pairing means $\mathbb{R}$-bilinear and nondegenerate.

1.2.19 Definition (Functional tensor and wedge products)
Let $X, Y, X_i \in \{\mathcal{V}^1, \mathcal{F}^1\}$, $i = 1, \ldots, r$.

(i) The *functional tensor product* $X \otimes Y := \mathrm{Hom}(X^*, Y)$ is the set of all $q \times q$-matrix differential operators $\mathcal{D} : X^* \to Y$.

(ii) The *functional tensor product* $\bigotimes_i X_i := \mathrm{Hom}(X_1^*, X_2^*, \ldots, X_{r-2}^*; X_{r-1} \otimes X_r)$ is the set of all $\mathbb{R}$-multilinear maps

$$\mathcal{D} : \begin{cases} X_1^* \times X_2^* \times \cdots \times X_{r-2}^* & \to \quad X_{r-1} \otimes X_r \\ (S_1, \ldots, S_{r-2}) & \mapsto \quad \mathcal{D}(S_1, \ldots, S_{r-2}) \end{cases}$$

of the form

$$\mathcal{D}(S_1, \ldots, S_{r-2}) = (a^{J_1 \ldots J_{r-2}, J}_{\alpha_1 \ldots \alpha_{r-2}, \alpha\beta} (S_1^{\alpha_1})_{J_1} \cdots (S_{r-2}^{\alpha_{r-2}})_{J_{r-2}} D_J)_{\alpha\beta},$$

where $(S_i^\gamma)_I := D_I(S_i^\gamma)$ are *total* derivatives and the coefficients[15] $a^{J_1 \ldots J_{r-2}, J}_{\alpha_1 \ldots \alpha_{r-2}, \alpha\beta}$ are jet expressions, which uniquely determine $\mathcal{D}$, but are also *uniquely*[16] determined by $\mathcal{D}$. In the following $\mathcal{D}$ is denoted by its image $\mathcal{D}(S_1, \ldots, S_{r-2})$ for general $S_1, \ldots, S_{r-2}$.

(iii) The *functional wedge product* $X^2 = \Lambda^2 X$ is the set of all skew-adjoint operators $\mathcal{D} \in X \otimes X$.

(iv) The *functional k-th wedge product* $X^k = \Lambda^k X$ is the set of all operators $\mathcal{D} \in \bigotimes^k X$ satisfying

  (a) $\mathcal{D}(S_1, \ldots, S_{k-2}) \in X^2$

  (b) $\mathcal{D}(S_1, \ldots, S_i, \ldots, S_{k-2}) S_{k-1} = -\mathcal{D}(S_1, \ldots, S_{k-1}, \ldots, S_{k-2}) S_i$

  for all $i = 1, \ldots, k-2$ and all $S_j \in X^*$, $j = 1, \ldots, k-1$.

The definition of the *functional k-th symmetric product* $S^k X$ is analogues. Elements of $\bigotimes^r \mathcal{V}^1 \otimes \bigotimes^s \mathcal{F}^1$ are called *functional* (r, s)*-tensors*. $\mathcal{V}^k = \Lambda^k \mathcal{V}^1$ (resp. $\mathcal{F}^k = \Lambda^k \mathcal{F}^1$) is called the space of *functional k-vectors* (resp. *functional k-forms*).

1.2.20 Remark
$X \otimes Y$ is not the classical tensor product, but is rather a tensor product over $\mathcal{F}^0$. This should not be interpreted in a classical manner either, since $\mathcal{F}^0$ has no multiplication structure ($p > 0$); functional spaces are *not* modules over $\mathcal{F}^0$.

[15] For the coefficients $a^{\because}_{\because}$ the multi-indices J, J_i's are upper indices. The position of α and β will be discussed in Remark 1.2.21. α_i is an upper (resp. lower) index, if $X_i^* \cong \mathcal{F}^1$ (resp. $X_i^* \cong \mathcal{V}^1$), i.e. it is always opposite to the position at the $S_i \in X_i^*$.

[16] See the discussion below Example 1.2.22.

1.2.21 Remark
The four spaces $\mathcal{V}^1\otimes\mathcal{V}^1$, $\mathcal{F}^1\otimes\mathcal{F}^1$, $\mathcal{F}^1\otimes\mathcal{V}^1$, and $\mathcal{V}^1\otimes\mathcal{F}^1$ are defined as the space of $q\times q$-matrix differential operators $\mathcal{D}:\mathcal{A}^q\to\mathcal{A}^q$. To apply such an operator, identify $\mathcal{F}^1$ resp. $\mathcal{V}^1$ with the q-tuple space $\mathcal{A}^q$ and apply the matrix operator from the left. Hence, as matrices one cannot optically distinguish the elements of these four spaces. For $\mathcal{D}\in\mathcal{V}^1\otimes\mathcal{V}^1$ (resp. $\mathcal{F}^1\otimes\mathcal{F}^1$, $\mathcal{F}^1\otimes\mathcal{V}^1$) one writes $\mathcal{D}=(\mathcal{D}^{\alpha\beta})$ (resp. $(\mathcal{D}_{\alpha\beta})$, $(\mathcal{D}^{\alpha}_{\beta})$) to distinguish the first three of them. Since the last two spaces are naturally isomorphic (cf. Corollary 2.8.4) there is no urgent[17] need to distinguish their elements (see Section 2.8.) As usual, composition is only possible if one sums over an upper and a lower index, e.g. two elements of $\mathcal{V}^1\otimes\mathcal{V}^1$ are not composable.

1.2.22 Example
For $T=(T^u,T^v)\in\mathcal{A}^2$ the operator[18]

$$\mathcal{D}=\mathcal{D}(T)=\begin{pmatrix} -2vT^vD_x-v_xT^v-vT^v_x & vT^uD_x+2vT^u_x+v_xT^u\\ vT^uD_x-vT^u_x & 0\end{pmatrix}$$

can be regarded as a functional (r,s)-tensor with $r+s=3$. If one specifically views it as a functional $(3,0)$-tensor, i.e. an element of $\mathcal{V}^1\otimes\mathcal{V}^1\otimes\mathcal{V}^1$, (resp. $(0,3)$-tensor, i.e. an element of $\mathcal{F}^1\otimes\mathcal{F}^1\otimes\mathcal{F}^1$), then one can immediately verify the *total* skew-adjointness properties appearing in (iv):

$$\mathcal{D}(T)^*=-\mathcal{D}(T),\ \mathcal{D}(T)S=-\mathcal{D}(S)T.$$

Thus $\mathcal{D}(T)$ is *even* a functional 3-vector, i.e. an element of $\mathcal{V}^3$ (resp. functional 3-form, i.e. an element of $\mathcal{F}^3$). Cf. Example 3.5.7, where the above operator appears naturally as a functional 3-form.
In contrast

$$\mathcal{E}(T)=\begin{pmatrix} 0 & -T^uT^v_xD_x\\ -T^vD_x-2T^v_x & 2T^uD_x+T^u_x\end{pmatrix}$$

is not linear in T and therefore *not* a functional tensor.

One can use the general integration by parts formula (1.8) to justify the definitions of functional spaces starting with more natural ones like

$$X\otimes Y:=\operatorname{Hom}(X^*,Y^*;\mathcal{F}^0),$$

where $\operatorname{Hom}(X^*,Y^*;\mathcal{F}^0)$ denotes the space of bilinear total differential operators of $X^*\times Y^*$ with values in $\mathcal{F}^0$, i.e. an operator $\mathcal{D}\in\operatorname{Hom}(X^*,Y^*;\mathcal{F}^0)$ depends linearly on the components of the first (resp. second) argument and its *total* derivatives and takes values in $\mathcal{F}^0$. More explicitly, $\mathcal{D}(S_1,S_2)=a^{J_1J_2}_{\alpha_1\alpha_2}(S_1^{\alpha_1})_{J_1}(S_2^{\alpha_2})_{J_2}$, where the coefficients $a^{J_1J_2}_{\alpha_1\alpha_2}$ are jet expressions uniquely determining $\mathcal{D}$, but are *not*

[17] By using $\mathcal{D}^{\alpha}{}_{\beta}$ and $\mathcal{D}_{\beta}{}^{\alpha}$ one can indeed distinguish them.
[18] $T^{\alpha}_J:=D_JT^{\alpha}$.

uniquely determined by $\mathcal{D}$. In this approach the space of functional k-forms $\mathcal{F}^k$ can be easily defined as a subspace of operators $\mathcal{D} \in \mathrm{Hom}(\underbrace{\mathcal{V}^1, \ldots, \mathcal{V}^1}_{k}; \mathcal{F}^0)$ satisfying

$$\mathcal{D}(S_1, \ldots, S_i, \ldots, S_j, \ldots, S_k) = -\mathcal{D}(S_1, \ldots, S_j, \ldots, S_i, \ldots, S_k),$$

for all $1 \leq i < j \leq k$ and all $S_l \in \mathcal{V}^1$, $l = 1, \ldots, k$.
Note that this an identity between functionals, which expresses the drawback of this "simpler" approach. One cannot directly check the vanishing of a functional, making it hard to construct useful normal forms for the functional spaces. The approach suggested in 1.2.19 eliminates this difficulty. As mentioned, one can use the integration by parts formula to prove the equivalence[19] of both definitions.

1.2.23 Definition (Contraction)
Let $X, X_1, \ldots, X_{r-1} \in \{\mathcal{V}^1, \mathcal{F}^1\}$ be functional spaces. A *contraction* is one of the following maps, which one denotes by $\langle \cdot, \cdot \rangle$:

(i) A *pairing* $\mathcal{F}^1 \times \mathcal{V}^1 \to \mathcal{F}^0$; $(\Delta, Q) \mapsto \Delta \cdot Q := \Delta_\alpha Q^\alpha$.

(ii) An *action*
$$\begin{cases} (X_1 \otimes \cdots \otimes X_{r-2} \otimes \mathcal{F}^1 \otimes X_{r-1}) \times \mathcal{V}^1 & \to & X_1 \otimes \cdots \otimes X_{r-1} \\ (\mathcal{D}(S_1, \ldots, S_{r-2}), R) & \mapsto & \mathcal{D}(S_1, \ldots, S_{r-2})R \end{cases}.$$

(iii) A *composition*
$$\begin{cases} (X_1 \otimes \cdots \otimes X_{r-2} \otimes \mathcal{F}^1 \otimes X_{r-1}) \times (X \otimes \mathcal{V}^1) & \to & X_1 \otimes \cdots \otimes X_{r-1} \otimes X \\ (\mathcal{D}(S_1, \ldots, S_{r-2}), \mathcal{R}) & \mapsto & \mathcal{D}(S_1, \ldots, S_{r-2})\mathcal{R} \end{cases}.$$

(iv) One of the above with $\mathcal{F}^1$ and $\mathcal{V}^1$ interchanged.

A contraction is thus a sum over lower and upper indices (for the multi-indices Formula (1.9) is used). Further, a contraction involves only *total* derivatives. Of course, one can also consider more general contractions with more than two arguments, or where one sums over more than one pair of upper and lower indices; these, however, do not occur in this thesis.

1.3 Basic Lemmas

We begin this section with the Fréchet derivative, which is the core part[20] of the vertical derivative mentioned in the introduction.

[19] One uses the integration by parts formula, to eliminate one multi-index: $(J_1 \ldots J_r) \to (J_1 \ldots J_{r-2}, J)$. This yields the uniqueness of the coefficients (cf. Definition 1.2.19, (ii)).

[20] In the language of the variational bicomplex: $\delta A = \frac{\partial A}{\partial u_J^\alpha} \delta u_J^\alpha = (\mathsf{D}_A)_\alpha \delta u^\alpha$ for $A \in \mathcal{A} \equiv \Omega^{0,0}$, with the contact forms $\delta u_J^\alpha := du_J^\alpha - u_{J+1_i}^\alpha dx^i$. The Fréchet derivative applied to (new) linearised dependent variables $(v^1, \ldots, v^q)$ yields the linearised equations for the vertical space $R_q = V(\mathcal{R}_q)$ in the language of [Pom], p. 83.

1.3.1 Definition (Fréchet derivative)
For $T \in \mathcal{A}^r$ the differential operator $\mathsf{D}_T : \mathcal{V}^1 \to \mathcal{A}^r$ defined by

$$\mathsf{D}_T = (\frac{\partial T}{\partial u_J^1} D_J, \dots, \frac{\partial T}{\partial u_J^q} D_J) \tag{1.12}$$

is called the *Fréchet derivative* of T.

1.3.2 Example

1. The Fréchet derivative of the characteristic $R = u_{xxx} + uu_x$ is

$$\mathsf{D}_R = D_{xxx} + uD_x + u_x.$$

2. The Fréchet derivative of the characteristic

$$Q = \begin{pmatrix} v_x - u_t \\ \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x - v_t \end{pmatrix}$$

is the 2×2-matrix differential operator

$$\mathsf{D}_Q = \begin{pmatrix} -D_t & D_x \\ \frac{1}{3}D_{xxx} + \frac{8}{3}uD_x + \frac{8}{3}u_x & -D_t \end{pmatrix}.$$

We first note the following two basic formulas. The first one relates the prolongation of an evolutionary vector field and the Fréchet derivative:

$$\mathsf{pr}\,\mathbf{v}_Q(T) = \mathsf{D}_T Q, \tag{1.13}$$

for all $T \in \mathcal{A}^r$ and $Q \in \mathcal{V}^1$. The proof follows immediately from the prolongation formula (1.5) and the definition of the Fréchet derivative (1.12). The second formula is the standard LEIBNIZ product rule

$$\mathsf{pr}\,\mathbf{v}(L \cdot P) = \mathsf{pr}\,\mathbf{v}L \cdot P + L \cdot \mathsf{pr}\,\mathbf{v}P \tag{1.14}$$

where $\mathbf{v}$ is a generalized vector field and L, P are arbitrary differential expressions. It just expresses the fact, that a vector field is a derivation.

1.3.3 Lemma ([Olv], Lemma 5.12)
For $P \in \mathcal{A}$ and $R \in \mathcal{V}^1$ the following commutation rule holds

$$\mathsf{pr}\,\mathbf{v}_R(D_i P) = D_i(\mathsf{pr}\,\mathbf{v}_R P). \tag{1.15}$$

PROOF. From the simple commutation relation

$$\frac{\partial}{\partial u_J^\alpha}(D_i P) = D_i(\frac{\partial P}{\partial u_J^\alpha}) + \frac{\partial P}{\partial u_{J-1_i}^\alpha},$$

we get

$$\begin{aligned}
\mathsf{pr}\,\mathbf{v}_R(D_iP) &\overset{(1.5)}{=} D_J(R^\alpha)\frac{\partial}{\partial u_J^\alpha}(D_iP)\\
&= D_J(R^\alpha)D_i(\frac{\partial P}{\partial u_J^\alpha}) + D_J(R^\alpha)\frac{\partial P}{\partial u_{J-1_i}^\alpha}\\
&= D_J(R^\alpha)D_i(\frac{\partial P}{\partial u_J^\alpha}) + D_iD_J(R^\alpha)\frac{\partial P}{\partial u_J^\alpha}\\
&= D_i(\mathsf{pr}\,\mathbf{v}_R P).
\end{aligned}$$

□

1.3.4 Definition (Directional derivative)
Let $\mathcal{D} = P^J D_J : \mathcal{A} \to \mathcal{A}$ be a differential operator and $Q \in \mathcal{V}^1$. The *directional derivative*[21] $\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})$ with respect to the evolutionary vector field $\mathbf{v}_Q$ is defined by

$$\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}) = \mathsf{pr}\,\mathbf{v}_Q(P^J)D_J. \tag{1.16}$$

The directional derivative of a matrix differential operator is the matrix of the directional derivatives of the entries.

1.3.5 Lemma
The directional derivative of a differential operator is defined in such a way, that the following Leibniz rule[22] *for operators holds:*

$$\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}T) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{pr}\,\mathbf{v}_Q(T), \tag{1.17}$$

or equivalently by (1.13)

$$\mathsf{D}_{\mathcal{D}T}(Q) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{D}_T Q, \tag{1.18}$$

for arbitrary differential operators $\mathcal{D} : \mathcal{A}^r \to \mathcal{A}^s$ and $T \in \mathcal{A}^r$. This property uniquely determines the directional derivative of a differential operator.

PROOF. This is a direct consequence of Formula (1.15) and the standard Leibniz rule (1.14). □

1.3.6 Example
For $R = u_{xxx} + uu_x$ and $\mathcal{E} := D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$ one verifies that

$$\mathsf{pr}\,\mathbf{v}_R(\mathcal{E}) = (\frac{2}{3}u_{xxx} + \frac{2}{3}uu_x)D_x + (\frac{1}{3}u_{xxxx} + \frac{1}{3}uu_{xx} + \frac{1}{3}u_x^2).$$

Using the calculations of Example 1.2.12 we have just verified

$$\mathsf{pr}\,\mathbf{v}_R(\mathcal{E}) - \mathsf{D}_R\mathcal{E} - \mathcal{E}\mathsf{D}_R^* = 0.$$

Cf. Theorem 4.2.4 and Example 4.3.13.

[21] I deliberately avoid calling $\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})$ a Lie derivative, as suggested in [Olv].
[22] [Olv], Formula (5.38).

1.3.7 Remark
If in the following the differential operator is not specified any further, then the statement is valid for the general case.

1.3.8 Corollary
For two composable differential operators $\mathcal{D}, \mathcal{E}$ we have the following Leibniz rule:

$$\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}\cdot\mathcal{E}) = \mathsf{pr}\,\mathbf{v}_Q\mathcal{D}\cdot\mathcal{E} + \mathcal{D}\cdot\mathsf{pr}\,\mathbf{v}_Q\mathcal{E}. \tag{1.19}$$

PROOF. Apply Formula (1.17) twice. □

1.3.9 Definition (Directional derivative of functional tensors)
Let $\mathcal{D}(S_1,\ldots,S_{r-2}) = (a^{J_1\ldots J_{r-2},J}_{\alpha_1\ldots\alpha_{r-2},\alpha\beta}(S_1^{\alpha_1})_{J_1}\cdots(S_{r-2}^{\alpha_{r-2}})_{J_{r-2}}D_J)_{\alpha\beta}$ be a functional r-tensor and $Q\in\mathcal{V}^1$. The *directional derivative*[23] $\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})$ is defined by

$$\begin{aligned}&\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})(S_1,\ldots,S_{r-2}) = \\ &\quad(\mathsf{pr}\,\mathbf{v}_Q(a^{J_1\ldots J_{r-2},J}_{\alpha_1\ldots\alpha_{r-2},\alpha\beta})(S_1^{\alpha_1})_{J_1}\cdots(S_{r-2}^{\alpha_{r-2}})_{J_{r-2}}D_J)_{\alpha\beta}.\end{aligned} \tag{1.20}$$

1.3.10 Lemma
For a differential operator $\mathcal{D}$ and $R\in\mathcal{V}^1$ the following commutation rule holds

$$\mathsf{pr}\,\mathbf{v}_R(\mathcal{D}^*) = \mathsf{pr}\,\mathbf{v}_R(\mathcal{D})^*. \tag{1.21}$$

PROOF. Without loss of generality $\mathcal{D} = P^J D_J$ ($P^J\in\mathcal{A}$). By using Lemma 1.3.5, we obtain

$$\begin{aligned}\mathsf{pr}\,\mathbf{v}_R(\mathcal{D}^*) &= \mathsf{pr}\,\mathbf{v}_R((-1)^{|I|}D_I(P^I\cdot)) \\ &\overset{(1.9)}{=} \sum_J(-1)^{|I|}\mathsf{pr}\,\mathbf{v}_R(\binom{I}{J}D_J(P^I)D_{I-J}) \\ &\overset{(1.15)}{=} \sum_J(-1)^{|I|}\binom{I}{J}D_J(\mathsf{pr}\,\mathbf{v}_R(P^I))D_{I-J} \\ &= (-1)^{|I|}D_I(\mathsf{pr}\,\mathbf{v}_R(P^I)\cdot) \\ &= \mathsf{pr}\,\mathbf{v}_R(\mathcal{D})^*.\end{aligned}$$

□

The following two lemmas are the key tools in this work. They describe the interaction between the prolongation operation and the Fréchet derivative. The first one is simpler and is a part of the proof of the second.

1.3.11 Lemma (First key lemma)
For $L\in\mathcal{A}$ and $R,S\in\mathcal{V}^1$ the following differential expressions coincide:

$$\mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L)S = \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_L)R. \tag{1.22}$$

[23] It can be viewed as a *trivial* Lie derivative.

Proof.

$$\begin{aligned}\mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L)S &= D_I(R^\alpha)\frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta}D_J(S^\beta)\\ &= \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_L)R.\end{aligned}$$

□

1.3.12 Lemma (Second key lemma)
For $L, P \in \mathcal{A}$ the operator[24] $\mathsf{pr}\,\mathbf{v}_.(\mathsf{D}_L^*)P : \mathcal{V}^1 \to \mathcal{F}^1; R \mapsto \mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L^*)P$ *is self-adjoint:*

$$(\mathsf{pr}\,\mathbf{v}_.(\mathsf{D}_L^*)P)^* = \mathsf{pr}\,\mathbf{v}_.(\mathsf{D}_L^*)P \tag{1.23}$$

Proof. Using the two previous lemmas, we get for arbitrary $R, S \in \mathcal{V}^1$ the following identities of *functionals*

$$\begin{aligned}\mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L^*)P\cdot S &\overset{(1.21)}{=} \mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L)^*P\cdot S\\ &\overset{(1.8)}{=} P\cdot\mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L)S\\ &\overset{(1.22)}{=} \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_L)R\cdot P\\ &\overset{(1.21)}{=} R\cdot\mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_L^*)P.\end{aligned}$$

□

As a corollary we get the following important formula:

1.3.13 Corollary ([Olv], Formula (5.60))
For $L \in \mathcal{A}$ and $R \in \mathcal{V}^1$

$$\mathsf{D}_{\mathsf{pr}\,\mathbf{v}_R(L)} = \mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L) + \mathsf{D}_L\mathsf{D}_R. \tag{1.24}$$

Proof. For an arbitrary $S \in \mathcal{V}^1$ we have

$$\begin{aligned}\mathsf{D}_{\mathsf{pr}\,\mathbf{v}_R(L)}S &\overset{(1.13)}{=} \mathsf{D}_{\mathsf{D}_L R}S\\ &\overset{(1.18)}{=} \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_L)R + \mathsf{D}_L\mathsf{D}_R S\\ &\overset{(1.22)}{=} \mathsf{pr}\,\mathbf{v}_R(\mathsf{D}_L)S + \mathsf{D}_L\mathsf{D}_R S.\end{aligned}$$

□

For those of the above results that also appear in [Olv], proofs are provided if they are omitted in the book, or if the proof given here is more simpler or more direct. The above definition of higher functional spaces and at least the second key lemma seem to be new.

[24] In [GDo1], p. 257, $\mathsf{pr}\,\mathbf{v}_.(\mathcal{H})$ is called the Fréchet derivative of the operator $\mathcal{H}$ and denoted by $\mathsf{D}_{\mathcal{H}}$: $(\mathsf{D}_{\mathcal{H}}\Delta)(Q) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{H})\Delta$.

Chapter 2

The Lie Derivative and Lie Module Structures

2.1 Leibniz Rule

2.1.1 Definition (Lie derivative)
For a generalized vector field $\mathbf{v}$ a *Lie derivative* $\mathcal{L}_{\mathbf{v}}$ satisfies the following properties[1]:

(i) The Lie derivative is $\mathbb{R}$-linear and preserves the type of functional tensors, i.e. if it is defined for a functional space X, then it carries elements of X to elements of X, more precisely $\mathcal{L}_{\mathbf{v}} : X \to X$.

(ii) If X, Y and Z are functional spaces, and $\langle\cdot,\cdot\rangle : X \times Y \to Z$ is a contraction, then the Lie derivative satisfies the *Leibniz rule*

$$\mathcal{L}_{\mathbf{v}}\langle a, b\rangle = \langle \mathcal{L}_{\mathbf{v}} a, b\rangle + \langle a, \mathcal{L}_{\mathbf{v}} b\rangle, \tag{2.1}$$

for all $a \in X$ and $b \in Y$.

(iii) For all functional spaces X for which a Lie derivative is defined, the Lie derivative defines an $\mathbb{R}$-bilinear map

$$\langle\cdot,\cdot\rangle : \left\{ \begin{array}{ccc} \mathcal{V}^1 \times X & \to & X \\ (R, a) & \mapsto & \mathcal{L}_{\mathbf{v}_R} a \end{array} \right. \tag{2.2}$$

satisfying (2.1). Let us call it a *non-total contraction*[2].

[1] Note, properties (i) and (ii) are also satisfied for covariant derivatives ∇_v. It is condition (iii) that enforces the *uniqueness* of the Lie derivative; it can also be written as $\mathcal{L}_{\mathbf{v}}(\mathcal{L}) = 0$.

[2] Although it is *not* a contraction in the sense of Definition 1.2.23. It may be regarded as a "generalized" contraction.

The philosophy of this chapter is to utilize the above properties of the Lie derivative, to *recursively* define a $\mathcal{V}^1$-Lie module structure for *all* functional tensor spaces, where the functional space $\mathcal{V}^1$ will be endowed with a Lie algebra structure.

2.1.2 Remark (Non-total contraction)
Combining (ii) and (iii) one gets $\mathfrak{L}_{\mathbf{v}}\mathfrak{L}_{\mathbf{v}_R}a = \mathfrak{L}_{\mathbf{v}}\langle R, a\rangle = \langle \mathfrak{L}_{\mathbf{v}}R, a\rangle + \langle R, \mathfrak{L}_{\mathbf{v}}a\rangle = \mathfrak{L}_{\mathbf{v}_{\mathfrak{L}_{\mathbf{v}}R}}\, a + \mathfrak{L}_{\mathbf{v}_R}\mathfrak{L}_{\mathbf{v}}a$. Hence for all $a \in X$

$$[\mathfrak{L}_{\mathbf{v}}, \mathfrak{L}_{\mathbf{v}_R}]\, a = \mathfrak{L}_{\mathbf{v}_{\mathfrak{L}_{\mathbf{v}}R}}\, a. \tag{2.3}$$

2.1.3 Lemma
For two generalized vector fields $\mathbf{v}$ *and* $\mathbf{w}$ *the* commutator $[\mathfrak{L}_{\mathbf{v}}, \mathfrak{L}_{\mathbf{w}}]$ *of the Lie derivatives* $\mathfrak{L}_{\mathbf{v}}$ *and* $\mathfrak{L}_{\mathbf{w}}$ *satisfies the above conditions (i) and (ii).*

PROOF. (i) is obvious. For (ii) take $a \in X$ and $b \in Y$ arbitrary. Then

$$\begin{aligned}
\langle \mathfrak{L}_{\mathbf{v}}\mathfrak{L}_{\mathbf{w}}a, b\rangle &= \langle \mathfrak{L}_{\mathbf{v}}(\mathfrak{L}_{\mathbf{w}}a), b\rangle \\
&\overset{(2.1)}{=} \mathfrak{L}_{\mathbf{v}}\langle \mathfrak{L}_{\mathbf{w}}a, b\rangle - \langle \mathfrak{L}_{\mathbf{w}}a, \mathfrak{L}_{\mathbf{v}}b\rangle \\
&\overset{(2.1)}{=} \mathfrak{L}_{\mathbf{v}}(\mathfrak{L}_{\mathbf{w}}\langle a, b\rangle - \langle a, \mathfrak{L}_{\mathbf{w}}b\rangle) - \langle \mathfrak{L}_{\mathbf{w}}a, \mathfrak{L}_{\mathbf{v}}b\rangle \\
&\overset{(2.1)}{=} \mathfrak{L}_{\mathbf{v}}\mathfrak{L}_{\mathbf{w}}\langle a, b\rangle - \langle a, \mathfrak{L}_{\mathbf{v}}\mathfrak{L}_{\mathbf{w}}b\rangle - \langle \mathfrak{L}_{\mathbf{v}}a, \mathfrak{L}_{\mathbf{w}}b\rangle - \langle \mathfrak{L}_{\mathbf{w}}a, \mathfrak{L}_{\mathbf{v}}b\rangle
\end{aligned}$$

and hence

$$\langle [\mathfrak{L}_{\mathbf{v}}, \mathfrak{L}_{\mathbf{w}}]a, b\rangle = [\mathfrak{L}_{\mathbf{v}}, \mathfrak{L}_{\mathbf{w}}]\langle a, b\rangle - \langle a, [\mathfrak{L}_{\mathbf{v}}, \mathfrak{L}_{\mathbf{w}}]b\rangle. \tag{2.4}$$

This is the elementary proof of the well-known fact, that the commutator of two derivations is again a derivation. □

2.2 Functionals

In this section the behaviour of functionals (functional 0-forms) under infinitesimal transformations is studied.

2.2.1 Lemma ([Olv], Formula (4.15))
A Lagrangian $L \in \mathcal{A}$ *transforms infinitesimally according to the rule*

$$\mathfrak{L}_{\mathbf{v}}L = \mathsf{pr}\,\mathbf{v}L + L\mathsf{Div}(\xi), \tag{2.5}$$

where $\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ *is a generalized vector field*[3] *and* $\mathfrak{L}_{\mathbf{v}}$ *is the* Lie derivative *with respect to* $\mathbf{v}$*.*

[3] [Olv] proves this for point vector fields only. His proof uses the fact, that L is an "integrand", or more precisely a horizontal p-form $Ldx^1 \wedge \ldots \wedge dx^p$, and the fact that a vector field is a derivation. The above Lie derivative coincides with the notion of projected Lie derivative $\mathfrak{L}^\sharp_{\mathbf{v}}$ for $(p, 0)$-forms (i.e. horizontal p-forms) introduced in [And], Chapter 3.

PROOF. [Olv], Theorem 4.12. □

2.2.2 Corollary (Lie derivative of functionals)
For a Lagrangian L viewed as an element of $\mathcal{F}^0$, i.e. as a functional, the Lie derivative $\mathcal{L}_{\mathbf{v}} : \mathcal{F}^0 \to \mathcal{F}^0$ with respect to a generalized vector field $\mathbf{v}$ is given by

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\, \mathbf{v}_Q L \tag{2.6}$$

where Q is the characteristic of $\mathbf{v}$. One also says that the functional L transforms infinitesimally *according to the above rule.*

PROOF.

$$\begin{aligned} \mathcal{L}_{\mathbf{v}} L &\overset{(2.5)}{=} \mathsf{pr}\, \mathbf{v} L + L \mathsf{Div}(\xi) \\ &\overset{(1.6)}{=} \mathsf{pr}\, \mathbf{v}_Q L + \xi^i D_i L + L D_i \xi^i \\ &= \mathsf{pr}\, \mathbf{v}_Q L + \mathsf{Div}(L\xi) \\ &= \mathsf{pr}\, \mathbf{v}_Q L, \end{aligned}$$

where the last equality is one between functionals. □

2.2.3 Corollary
For a Lagrangian L viewed as an element of $\mathcal{F}^0$, the Lie derivatives $\mathcal{L}_{\mathbf{v}}$ and $\mathcal{L}_{\mathbf{v}_Q}$ coincide.

2.2.4 Axiom
We declare Formula (2.6) an axiom. *It is the starting point to* recursively *determine the Lie derivative, or algebraically speaking, the $\mathcal{V}^1$-Lie module structure for all functional tensor spaces.*

2.2.5 Definition (Variational symmetry)
For a Lagrangian $L \in \mathcal{A}$ a generalized vector field $\mathbf{v} = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}$ is called a *variational symmetry*, if

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\, \mathbf{v} L + L \mathsf{Div}(\xi) = 0 \tag{2.7}$$

in $\mathcal{A}$.

2.2.6 Definition (Bessel-Hagen symmetry)
For a functional $L \in \mathcal{F}^0$ a generalized vector field $\mathbf{v}$ with characteristic Q is called a *Bessel-Hagen symmetry* or *divergence symmetry*, if

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\, \mathbf{v}_Q L = 0 \tag{2.8}$$

in $\mathcal{F}^0$, i.e. if there exists a current $B \in \mathcal{A}^p$, such that

$$\mathsf{pr}\, \mathbf{v}_Q L = \mathsf{Div}(B) \tag{2.9}$$

in $\mathcal{A}$.

2.2.7 Corollary
The Bessel-Hagen symmetries of a functional $L \in \mathcal{F}^0$ form a Lie subalgebra of the Lie algebra of generalized vector fields.

Proof. This will follow from Formula (2.17). □

2.3 Characteristics

In this section we study the behaviour of characteristics (functional 1-vectors) under infinitesimal transformations.

2.3.1 Proposition ([Olv], Proposition 5.15)
(a) The commutator of two prolonged evolutionary vector fields is again a prolonged evolutionary vector field. More precisely,

$$[\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] = \mathsf{pr}\,\mathbf{v}_{Q_3}, \tag{2.10}$$

with

$$Q_3 = \mathsf{pr}\,\mathbf{v}_{Q_1} Q_2 - \mathsf{pr}\,\mathbf{v}_{Q_2} Q_1. \tag{2.11}$$

(b) The commutator of two prolonged generalized vector fields $\mathsf{pr}\,\mathbf{v}_1, \mathsf{pr}\,\mathbf{v}_2$ is again a prolonged generalized vector field $\mathsf{pr}\,\mathbf{v}_3$, with

$$\mathbf{v}_3 = \{\mathsf{pr}\,\mathbf{v}_1(\xi_2^i) - \mathsf{pr}\,\mathbf{v}_2(\xi_1^i)\}\frac{\partial}{\partial x^i} + \{\mathsf{pr}\,\mathbf{v}_1(\eta_2^\alpha) - \mathsf{pr}\,\mathbf{v}_2(\eta_1^\alpha)\}\frac{\partial}{\partial u^\alpha}.$$

If Q_1 (resp. Q_2) is the characteristic of $\mathbf{v}_1$ (resp. $\mathbf{v}_2$), then $\mathbf{v}_3$ has the characteristic Q_3 given by the above formula.

Proof. (a) Let L be an arbitrary differential function, then

$$\begin{aligned}
[\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}](L) &= \mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{pr}\,\mathbf{v}_{Q_2} L) - \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{pr}\,\mathbf{v}_{Q_1} L) \\
&\overset{(1.13)}{=} \mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{D}_L Q_2) - \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{D}_L Q_1) \\
&\overset{(1.17)}{=} \mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{D}_L) Q_2 + \mathsf{D}_L \mathsf{pr}\,\mathbf{v}_{Q_1} Q_2 \\
&\quad - \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{D}_L) Q_1 - \mathsf{D}_L \mathsf{pr}\,\mathbf{v}_{Q_2} Q_1 \\
&\overset{(1.22)}{=} \mathsf{D}_L \mathsf{pr}\,\mathbf{v}_{Q_1} Q_2 - \mathsf{D}_L \mathsf{pr}\,\mathbf{v}_{Q_2} Q_1 \\
&\overset{(1.13)}{=} \mathsf{pr}\,\mathbf{v}_{\mathsf{pr}\,\mathbf{v}_{Q_1} Q_2 - \mathsf{pr}\,\mathbf{v}_{Q_2} Q_1} L.
\end{aligned}$$

(b) follows from (a) and Formula (1.6). □

2.3.2 Remark
The above proposition enables us to define the *Lie bracket* directly on evolutionary (resp. generalized) vector fields

$$[\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}] = \mathbf{v}_{Q_3} \quad (\text{resp. } [\mathbf{v}_1, \mathbf{v}_2] = \mathbf{v}_3),$$

with Q_3 (resp. $\mathbf{v}_3$) as above.

2.3.3 Theorem (Leibniz rule)
For every generalized vector field $\mathbf{v}$ *there exists one and only one operator* $\mathcal{L}_{\mathbf{v}} : \mathcal{V}^1 \to \mathcal{V}^1$ *satisfying the* Leibniz rule

$$\mathcal{L}_{\mathbf{v}}\langle R, L\rangle = \langle \mathcal{L}_{\mathbf{v}} R, L\rangle + \langle R, \mathcal{L}_{\mathbf{v}} L\rangle, \tag{2.12}$$

for all $R \in \mathcal{V}^1$ *and* $L \in \mathcal{F}^0$, *where*

$$\langle R, L\rangle := \mathcal{L}_{\mathbf{v}_R} L. \tag{2.13}$$

PROOF. Using the reformulated Leibniz rule (2.3) for the non-total contraction (2.13) one obtains

$$\mathsf{pr}\,\mathbf{v}_{\mathcal{L}_{\mathbf{v}} R} \overset{(2.6)}{=} \mathcal{L}_{\mathbf{v}_{\mathcal{L}_{\mathbf{v}} R}} \overset{(2.3)}{=} [\mathcal{L}_{\mathbf{v}}, \mathcal{L}_{\mathbf{v}_R}] \overset{(2.6)}{=} [\mathsf{pr}\,\mathbf{v}_Q, \mathsf{pr}\,\mathbf{v}_R], \tag{2.14}$$

where Q is the characteristic of $\mathbf{v}$. Finally the two formulas (2.10) and (2.11) yield $\mathcal{L}_{\mathbf{v}} R = \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{pr}\,\mathbf{v}_R Q$. □

2.3.4 Definition (Lie derivative of functional 1-vectors)
Let $\mathbf{v}$ be a generalized vector field with characteristic Q, and R an arbitrary characteristic, i.e. $R \in \mathcal{V}^1$. The Lie derivative of R with respect to $\mathbf{v}$ is given by

$$\begin{aligned} \mathcal{L}_{\mathbf{v}} R &= \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{pr}\,\mathbf{v}_R Q \\ &\overset{(1.13)}{=} \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{D}_Q R. \end{aligned} \tag{2.15}$$

If one defines[4] $\mathcal{L}_{\mathbf{v}} \mathbf{v}_R := \mathbf{v}_{\mathcal{L}_{\mathbf{v}} R}$ then by (2.10), (2.11) and Remark 2.3.2

$$\mathcal{L}_{\mathbf{v}} \mathbf{v}_R = [\mathbf{v}_Q, \mathbf{v}_R]. \tag{2.16}$$

One also says that the characteristic R (resp. evolutionary vector field $\mathbf{v}_R$) *transforms infinitesimally* according to the above rule.

2.3.5 Corollary
On $\mathcal{V}^1$, *both Lie derivatives* $\mathcal{L}_{\mathbf{v}}$ *and* $\mathcal{L}_{\mathbf{v}_Q}$ *coincide.*

PROOF. Deliberately, the proof given here does not make use of the special form of $\mathcal{L}_{\mathbf{v}}$ on $\mathcal{F}^0$, but solely of the obvious fact that the map $Q \mapsto \mathcal{L}_{\mathbf{v}_Q}$ on $\mathcal{F}^0$ is injective. As in Formula (2.14) one obtains $\mathcal{L}_{\mathbf{v}_{\mathcal{L}_{\mathbf{v}} R}} \overset{(2.3)}{=} [\mathcal{L}_{\mathbf{v}}, \mathcal{L}_{\mathbf{v}_R}] \overset{2.2.3}{=} [\mathcal{L}_{\mathbf{v}_Q}, \mathcal{L}_{\mathbf{v}_R}] \overset{(2.3)}{=} \mathcal{L}_{\mathbf{v}_{\mathcal{L}_{\mathbf{v}_Q} R}}$. By the mentioned injectivity the proof is done. □

2.3.6 Remark
The above results show that on $\mathcal{F}^0$, and therefore on $\mathcal{V}^1$, the Lie derivatives $\mathcal{L}_{\mathbf{v}}$ and $\mathcal{L}_{\mathbf{v}_Q}$ coincide. Out of these two spaces we constructed all functional spaces, thus both Lie derivatives will coincide for all these spaces. So we should *omit* $\mathcal{L}_{\mathbf{v}}$ in future. For the case of functional spaces $\mathcal{F}^s$ this fact is proved in [And], Chapter 3: $\mathcal{L}^\natural_X = \mathcal{L}^\natural_{X_{\mathrm{ev}}}$.

[4] Here $\mathbf{v}_R$ is viewed as an *evolutionary* and not merely as a generalized vector field.

2.3.7 Corollary
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q} = \mathsf{pr}\,\mathbf{v}_Q$ *on* $\mathcal{F}^0$ *satisfies the following rule:*

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]. \tag{2.17}$$

In light of the next proposition, this asserts that Lie derivative is a Lie algebra homomorphism, turning $\mathcal{F}^0$ *into a module for the Lie algebra* $\mathcal{V}^1$.

PROOF. The proof is done by (2.16) and (2.3) for $X = \mathcal{F}^0$. □

As required by (2.3), the Lie derivative will turn out to be a Lie algebra homomorphism for all functional spaces. This is the infinitesimal version of the associativity of coordinate changes.

2.3.8 Proposition
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q} = \mathsf{pr}\,\mathbf{v}_Q - \mathsf{D}_Q$ *on* $\mathcal{V}^1$ *satisfies the following rule:*

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]. \tag{2.18}$$

This is precisely the Jacobi identity for the Lie bracket

$$[[\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}], \mathbf{v}_{Q_3}] = [\mathbf{v}_{Q_1}, [\mathbf{v}_{Q_2}, \mathbf{v}_{Q_3}]] - [\mathbf{v}_{Q_2}, [\mathbf{v}_{Q_1}, \mathbf{v}_{Q_3}]],$$

and Q_3 *being an arbitrary characteristic. Hence, the Lie bracket turns* $\mathcal{V}^1$ *into a* Lie algebra. *The Lie derivative* $\mathcal{L}$ *becomes a Lie algebra homomorphism.*

PROOF. For the *non-degenerate*[5] contraction (2.13)

$$\begin{array}{rcl} \mathcal{V}^1 \times \mathcal{F}^0 & \to & \mathcal{F}^0 \\ (R, L) & \mapsto & \langle R, L\rangle \end{array}$$

one verifies

$$\begin{aligned} \langle [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]R, L\rangle &\overset{(2.4),(2.12)}{=} [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]\langle R, L\rangle - \langle R, [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]L\rangle \\ &\overset{(2.17)}{=} \mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}\langle R, L\rangle - \langle R, \mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}L\rangle \\ &\overset{(2.12)}{=} \langle \mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}R, L\rangle. \end{aligned}$$

There is of course a direct proof, which does not make use of the more *elegant* Leibniz principle for Lie derivatives. Let $\mathsf{pr}\,\mathbf{v}_Q = [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}]$, then

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_Q} &\overset{(2.15)}{=} \mathsf{pr}\,\mathbf{v}_Q - \mathsf{D}_Q \\ &\overset{(2.11)}{=} [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] - \mathsf{D}_{\mathsf{pr}\,\mathbf{v}_{Q_1} Q_2 - \mathsf{pr}\,\mathbf{v}_{Q_2} Q_1} \end{aligned}$$

[5] We make use of this fact for the first argument only.

$$
\begin{aligned}
&\overset{(1.24)}{=} && [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] - \mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{D}_{Q_2}) - \mathsf{D}_{Q_2}\mathsf{D}_{Q_1} + \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{D}_{Q_1}) + \mathsf{D}_{Q_1}\mathsf{D}_{Q_2} \\
&\overset{(1.17)}{=} && [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] - \mathsf{pr}\,\mathbf{v}_{Q_1}\mathsf{D}_{Q_2} + \mathsf{D}_{Q_2}\mathsf{pr}\,\mathbf{v}_{Q_1} - \mathsf{D}_{Q_2}\mathsf{D}_{Q_1} \\
& && + \mathsf{pr}\,\mathbf{v}_{Q_2}\mathsf{D}_{Q_1} - \mathsf{D}_{Q_1}\mathsf{pr}\,\mathbf{v}_{Q_2} + \mathsf{D}_{Q_1}\mathsf{D}_{Q_2} \\
&= && [\mathsf{pr}\,\mathbf{v}_{Q_1} - \mathsf{D}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2} - \mathsf{D}_{Q_2}] \\
&= && [\mathfrak{L}_{\mathbf{v}_{Q_1}}, \mathfrak{L}_{\mathbf{v}_{Q_2}}].
\end{aligned}
$$

□

For every formula of this type, there are two different proofs. The first one uses the *Leibniz principle*, emphasising the *axiomatic* approach followed in this thesis. The second proof uses the calculus developed in Chapter 1 and is given here to familiarise the reader with that calculus. This is done one further time (cf. Proposition 2.4.4).

2.3.9 Lemma
The map $\mathbf{v} \mapsto \mathbf{v}_Q$ is a Lie algebra homomorphism from the Lie algebra of generalized vector fields onto $\mathcal{V}^1$. The kernel consists of all total vector fields, *i.e. vector fields of the form $\xi^i D_i$, with $\xi^i \in \mathcal{A}$ $(i = 1, \ldots, p)$.*

Proof. The is precisely the last statement of (b) in Proposition 2.3.1. □

2.3.10 Definition (Evolutionary symmetry)
Let $P \in \mathcal{V}^1$ and $u_t = P$ be an evolution equation. $Q \in \mathcal{V}^1$ is called *evolutionary symmetry* of $u_t = P$, if

$$\mathfrak{L}_{\mathbf{v}_Q} P = 0. \tag{2.19}$$

2.3.11 Corollary
The evolutionary symmetries of an evolution equation $u_t = P$ form a Lie subalgebra of the Lie algebra of generalized symmetries of the evolution equation.

Proof. This follows from Proposition 2.3.8. □

The following easy lemma will be used to proof a criterion for recursion operators (Corollary 4.3.3).

2.3.12 Lemma (Symmetries of evolution equations)
Let $P \in \mathcal{V}^1$. The evolutionary vector field $\mathbf{v}_Q$ is a symmetry of the evolution equation[6] $u_t = P$, if and only if

$$\mathfrak{L}_{\mathbf{v}_P} Q = 0. \tag{2.20}$$

Proof. The proof is done by $\mathfrak{L}_{\mathbf{v}_P} Q = -\mathfrak{L}_{\mathbf{v}_Q} P$ and the definition of evolutionary symmetries. □

See also [Olv] Proposition 5.19, for a time dependent version.

[6] Neither P nor Q depend on t or any time derivative of $u = (u^1, \ldots, u^q)$.

2.4 Source Forms

In this section the behaviour of functional 1-forms under infinitesimal transformations is studied. Functional 1-forms are also called *source forms*[7].

2.4.1 Theorem (Leibniz rule)
For every evolutionary vector field $\mathbf{v}_Q$ *there exists one and only one operator* $\mathfrak{L}_{\mathbf{v}_Q} : \mathcal{F}^1 \to \mathcal{F}^1$ *satisfying the* Leibniz rule

$$\mathfrak{L}_{\mathbf{v}_Q}(\Delta \cdot R) = \mathfrak{L}_{\mathbf{v}_Q}\Delta \cdot R + \Delta \cdot \mathfrak{L}_{\mathbf{v}_Q}R. \tag{2.21}$$

for all $\Delta \in \mathcal{F}^1$ *and* $R \in \mathcal{V}^1$, *where*

$$\Delta \cdot R = \Delta_\alpha R^\alpha. \tag{2.22}$$

PROOF. The claim follows from the following equalities of functionals:

$$\begin{aligned}
&\mathfrak{L}_{\mathbf{v}_Q}(\Delta \cdot R) - \Delta \cdot \mathfrak{L}_{\mathbf{v}_Q}R \\
&\quad\overset{(2.6)}{=} \quad \mathsf{pr}\,\mathbf{v}_Q(\Delta \cdot R) - \Delta \cdot \mathfrak{L}_{\mathbf{v}_Q}R \\
&\quad\overset{(1.14),(2.15)}{=} \quad \mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{pr}\,\mathbf{v}_Q R - \Delta \cdot (\mathsf{pr}\,\mathbf{v}_Q R - \mathsf{D}_Q R) \\
&\quad= \quad \mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{D}_Q R \\
&\quad= \quad (\mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta) \cdot R.
\end{aligned}$$

Using the non-degeneracy of the pairing, one deduces $\mathfrak{L}_{\mathbf{v}_Q}\Delta = \mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta$. Here we observe the phenomenon, that precisely the non-total derivative $\mathsf{pr}\,\mathbf{v}_Q R$ of R cancels out. □

2.4.2 Definition (Lie derivative of functional 1-forms)
The Lie derivative of a source form $\Delta \in \mathcal{F}^1$ with respect to an evolutionary vector field $\mathbf{v}_Q$ is given by

$$\mathfrak{L}_{\mathbf{v}_Q}\Delta = \mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta. \tag{2.23}$$

One also says that the source form Δ *transforms infinitesimally* according to the above rule.

2.4.3 Remark
The identity of functionals

$$\mathfrak{L}_{\mathbf{v}_Q}\Delta \cdot R = \mathsf{pr}\,\mathbf{v}_Q\Delta \cdot R + \Delta \cdot \mathsf{pr}\,\mathbf{v}_R Q, \tag{2.24}$$

appears as formula (4.2) in [GDo2]. It is seen by (1.13) to coincide with the second last formula in the above proof.

[7]This notion is due to Takens.

2.4.4 Proposition
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q} = \mathsf{pr}\,\mathbf{v}_Q + \mathsf{D}^*_Q$ on $\mathcal{F}^1$ is a Lie algebra homomorphism, i.e.

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}], \tag{2.25}$$

turning $\mathcal{F}^1$ into a module for the Lie algebra $\mathcal{V}^1$.

Proof. For the *non-degenerate* contraction

$$\begin{array}{rcl} \mathcal{F}^1 \times \mathcal{V}^1 & \to & \mathcal{F}^0 \\ (\Delta, S) & \mapsto & \Delta \cdot S \end{array}$$

one verifies

$$\begin{aligned} ([\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]\Delta) \cdot S &\overset{(2.4),(2.21)}{=} [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}](\Delta \cdot S) - \Delta \cdot [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]S \\ &\overset{(2.17),(2.18)}{=} \mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}(\Delta \cdot S) - \Delta \cdot \mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}S \\ &\overset{(2.21)}{=} (\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]}\Delta) \cdot S. \end{aligned}$$

Again there is a direct proof, analogues to the second proof of 2.3.8. Let $\mathsf{pr}\,\mathbf{v}_{Q_3} = [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}]$, then

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{Q_3}} &\overset{(2.23)}{=} \mathsf{pr}\,\mathbf{v}_{Q_3} + \mathsf{D}^*_{Q_3} \\ &\overset{(2.11)}{=} [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] + \mathsf{D}^*_{\mathsf{pr}\,\mathbf{v}_{Q_1}Q_2 - \mathsf{pr}\,\mathbf{v}_{Q_2}Q_1} \\ &\overset{(1.24)}{=} [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] + (\mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{D}_{Q_2}) + \mathsf{D}_{Q_2}\mathsf{D}_{Q_1} - \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{D}_{Q_1}) - \mathsf{D}_{Q_1}\mathsf{D}_{Q_2})^* \\ &\overset{(1.21)}{=} [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] + \mathsf{pr}\,\mathbf{v}_{Q_1}(\mathsf{D}^*_{Q_2}) + \mathsf{D}^*_{Q_1}\mathsf{D}^*_{Q_2} - \mathsf{pr}\,\mathbf{v}_{Q_2}(\mathsf{D}^*_{Q_1}) - \mathsf{D}^*_{Q_2}\mathsf{D}^*_{Q_1} \\ &\overset{(1.17)}{=} [\mathsf{pr}\,\mathbf{v}_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2}] + \mathsf{pr}\,\mathbf{v}_{Q_1}\mathsf{D}^*_{Q_2} - \mathsf{D}^*_{Q_2}\mathsf{pr}\,\mathbf{v}_{Q_1} + \mathsf{D}^*_{Q_1}\mathsf{D}^*_{Q_2} \\ &\quad\; - \mathsf{pr}\,\mathbf{v}_{Q_2}\mathsf{D}^*_{Q_1} + \mathsf{D}^*_{Q_1}\mathsf{pr}\,\mathbf{v}_{Q_2} - \mathsf{D}^*_{Q_2}\mathsf{D}^*_{Q_1} \\ &= [\mathsf{pr}\,\mathbf{v}_{Q_1} + \mathsf{D}^*_{Q_1}, \mathsf{pr}\,\mathbf{v}_{Q_2} + \mathsf{D}^*_{Q_2}] \\ &= [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}]. \end{aligned}$$

□

2.4.5 Definition (Distinguished symmetry)
Let Δ be a source form, i.e. $\Delta \in \mathcal{F}^1$. An evolutionary vector field $\mathbf{v}_Q$ is called a *distinguished symmetry*[8] of Δ if

$$\mathcal{L}_{\mathbf{v}_Q}\Delta = 0.$$

[8] Following [And].

2.4.6 Corollary
The distinguished symmetries of a source form Δ form a Lie subalgebra of the Lie algebra of generalized symmetries of the source equation.

PROOF. This follows from Proposition 2.4.4. □

2.4.7 Definition (Generator of a local conservation law)
For $\Delta \in \mathcal{F}^1$ the characteristic $Q \in \mathcal{V}^1$ is called *generator of a (differential) local conservation law* of the source equation $\Delta = 0$, if there *locally* exists a current $P \in \mathcal{A}^p$, such that $Q \cdot \Delta = \mathsf{Div} P$. One calls P the *conserved current.*

2.4.8 Corollary
The Lie algebra of distinguished vector fields acts on the vector space of generators of local conservation laws.

PROOF. $Q \cdot \Delta = \mathsf{Div} P$ re-expressed in $\mathcal{F}^0$ becomes $Q \cdot \Delta = 0$. The proof is done by the Leibniz rule (2.21). □

By this one deduces easily

2.4.9 Corollary
The subspace of all distinguished symmetries $\mathbf{v}_Q$ of a source form Δ, where Q is a generator of a local conservation law of the source equation $\Delta = 0$, is a Lie ideal of the Lie algebra of distinguished symmetries of Δ.

2.5 General Functional Tensors

2.5.1 Theorem (Leibniz rule)
Let $X_1, \ldots, X_r \in \{\mathcal{V}^1, \mathcal{F}^1\}$ and $Z = X_1 \otimes \cdots \otimes X_r$. For every evolutionary vector field $\mathbf{v}_Q$ there exists one and only one operator $\mathfrak{L}_{\mathbf{v}_Q} : Z \to Z$ satisfying the Leibniz rule

$$\begin{aligned}\mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1}) &= \mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} \\ &+\mathcal{D}(\mathfrak{L}_{\mathbf{v}_Q} S_1, \ldots, S_{r-2})S_{r-1} + \cdots + \mathcal{D}(S_1, \ldots, \mathfrak{L}_{\mathbf{v}_Q} S_{r-2})S_{r-1} \\ &+\mathcal{D}(S_1, \ldots, S_{r-2})\mathfrak{L}_{\mathbf{v}_Q} S_{r-1}\end{aligned} \tag{2.26}$$

in X_r, or equivalently

$$\begin{aligned}\mathfrak{L}_{\mathbf{v}_Q}(S_r \cdot \mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1}) &= S_r \cdot \mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} \\ &+S_r \cdot \mathcal{D}(\mathfrak{L}_{\mathbf{v}_Q} S_1, \ldots, S_{r-2})S_{r-1} + \cdots + S_r \cdot \mathcal{D}(S_1, \ldots, \mathfrak{L}_{\mathbf{v}_Q} S_{r-2})S_{r-1} \\ &+S_r \cdot \mathcal{D}(S_1, \ldots, S_{r-2})\mathfrak{L}_{\mathbf{v}_Q} S_{r-1} + \mathfrak{L}_{\mathbf{v}_Q} S_r \cdot \mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1}\end{aligned} \tag{2.27}$$

in[9] *$\mathcal{F}^0$, for all $S_i \in X_i^*$.*

[9] This is an identity of functionals.

PROOF. The Leibniz rules (2.26) and (2.27) are easily seen to be equivalent by (2.21). Due to (2.15) and (2.23) the Lie derivative $\mathfrak{L}_{\mathbf{v}_Q} S_i = \mathsf{pr}\,\mathbf{v}_Q S_i + A_i S_i$, where $A_i := -\mathsf{D}_Q$ or $A_i := \mathsf{D}_Q^*$ depending on whether $X_i^* \cong \mathcal{V}^1$ or $X_i^* \cong \mathcal{F}^1$, for all $i = 1, \ldots, r-1$. $A_r := -\mathsf{D}_Q$ or $A_r := \mathsf{D}_Q^*$ depending on whether $X_r \cong \mathcal{V}^1$ or $X_r \cong \mathcal{F}^1$.

$$
\begin{aligned}
&\mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1}) - \mathcal{D}(S_1, \ldots, S_{r-2})\mathfrak{L}_{\mathbf{v}_Q} S_{r-1} \\
&\qquad -\mathcal{D}(\mathfrak{L}_{\mathbf{v}_Q} S_1, \ldots, S_{r-2})S_{r-1} - \cdots - \mathcal{D}(S_1, \ldots, \mathfrak{L}_{\mathbf{v}_Q} S_{r-2})S_{r-1} \\
&= \quad \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1}) + A_r\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad -\mathcal{D}(S_1, \ldots, S_{r-2})\mathsf{pr}\,\mathbf{v}_Q S_{r-1} - \mathcal{D}(S_1, \ldots, S_{r-2})A_{r-1}S_{r-1} \\
&\qquad -\mathcal{D}(\mathsf{pr}\,\mathbf{v}_Q S_1, \ldots, S_{r-2})S_{r-1} - \mathcal{D}(A_1 S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad - \cdots \\
&\qquad -\mathcal{D}(S_1, \ldots, \mathsf{pr}\,\mathbf{v}_Q S_{r-2})S_{r-1} - \mathcal{D}(S_1, \ldots, A_{r-2}S_{r-2})S_{r-1} \\
&\overset{(1.17)}{=} \quad \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} + A_r\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad +\mathcal{D}(\mathsf{pr}\,\mathbf{v}_Q S_1, \ldots, S_{r-2})S_{r-1} + \cdots + \mathcal{D}(S_1, \ldots, S_{r-2})\mathsf{pr}\,\mathbf{v}_Q S_{r-1} \\
&\qquad -\mathcal{D}(S_1, \ldots, S_{r-2})\mathsf{pr}\,\mathbf{v}_Q S_{r-1} - \mathcal{D}(S_1, \ldots, S_{r-2})A_{r-1}S_{r-1} \\
&\qquad -\mathcal{D}(\mathsf{pr}\,\mathbf{v}_Q S_1, \ldots, S_{r-2})S_{r-1} - \mathcal{D}(A_1 S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad - \cdots \\
&\qquad -\mathcal{D}(S_1, \ldots, \mathsf{pr}\,\mathbf{v}_Q S_{r-2})S_{r-1} - \mathcal{D}(S_1, \ldots, A_{r-2}S_{r-2})S_{r-1} \\
&= \quad \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} + A_r\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad -\mathcal{D}(A_1 S_1, \ldots, S_{r-2})S_{r-1} - \cdots - \mathcal{D}(S_1, \ldots, S_{r-2})A_{r-1}S_{r-1}.
\end{aligned}
\tag{2.28}
$$

Hence $\mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} = (2.28)$. Again we observe the phenomenon, that precisely the non-total derivative $\mathsf{pr}\,\mathbf{v}_Q S_i$ of S_i $(i = 1, \ldots, r-1)$ cancel out. □

2.5.2 Definition (Lie derivative of general functional tensors)
Let $X_1, \ldots, X_r \in \{\mathcal{V}^1, \mathcal{F}^1\}$, then the Lie derivative of a functional tensor $\mathcal{D} \in X_1^* \otimes \cdots \otimes X_{r-1}^* \otimes X_r \cong \mathrm{Hom}(X_1, \ldots, X_{r-2}; \mathrm{Hom}(X_{r-1}, X_r))$ with respect to an evolutionary vector field $\mathbf{v}_Q$ is given by

$$
\begin{aligned}
&\mathfrak{L}_{\mathbf{v}_Q}(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} \\
&\quad = \quad \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})(S_1, \ldots, S_{r-2})S_{r-1} + A_r\mathcal{D}(S_1, \ldots, S_{r-2})S_{r-1} \\
&\qquad\quad -\mathcal{D}(A_1 S_1, \ldots, S_{r-2})S_{r-1} - \cdots - \mathcal{D}(S_1, \ldots, S_{r-2})A_{r-1}S_{r-1},
\end{aligned}
\tag{2.29}
$$

where $A_i := -\mathsf{D}_Q$ or $A_i := \mathsf{D}_Q^*$ depending on whether $X_i \cong \mathcal{V}^1$ or $X_i \cong \mathcal{F}^1$, for all $i = 1, \ldots, r$. One also says that the functional tensor $\mathcal{D}$ *transforms infinitesimally* according to the above rule.

2.5.3 Proposition
The Lie derivative $\mathfrak{L} : \mathbf{v}_Q \mapsto \mathfrak{L}_{\mathbf{v}_Q}$ on $Z = X_1 \otimes \cdots \otimes X_r$ is a Lie algebra

homomorphism, i.e.

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}], \tag{2.30}$$

turning Z into a module for the Lie algebra $\mathcal{V}^1$.

PROOF. This is a straightforward generalization of Proposition 2.6.2 and its proof. □

2.6 Functional 2-Forms

2.6.1 Corollary (Lie derivative of functional 2-forms)
The Lie derivative of an operator $\mathcal{K} \in \mathcal{F}^1 \otimes \mathcal{F}^1$, i.e. an operator $\mathcal{K} : \mathcal{V}^1 \to \mathcal{F}^1$, especially a functional 2-form $\mathcal{K} \in \mathcal{F}^2$, with respect to the evolutionary vector field $\mathbf{v}_Q$ is given by

$$\mathcal{L}_{\mathbf{v}_Q}\mathcal{K} = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{K}) + \mathsf{D}_Q^*\mathcal{K} + \mathcal{K}\mathsf{D}_Q. \tag{2.31}$$

2.6.2 Proposition
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q}$ on $\mathcal{F}^1 \otimes \mathcal{F}^1$ is a Lie algebra homomorphism, i.e.

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}], \tag{2.32}$$

turning $\mathcal{F}^1 \otimes \mathcal{F}^1$ into a module for the Lie algebra $\mathcal{V}^1$.

PROOF. One reproduces the first proof of Proposition 2.4.4 using the *non-degenerate* contraction

$$\begin{array}{rcl} (\mathcal{F}^1 \otimes \mathcal{F}^1) \times \mathcal{V}^1 & \to & \mathcal{F}^1 \\ (\mathcal{K}, S) & \mapsto & \mathcal{K}S, \end{array}$$

and replacing (2.17) by (2.25), and (2.21) by (2.26). □

2.6.3 Corollary (Naturality of the adjoint operation)
The Lie derivative commutes with taking adjoints, i.e. for all $Q \in \mathcal{V}^1$

$$\mathcal{L}_{\mathbf{v}_Q}(\mathcal{K}^*) = \mathcal{L}_{\mathbf{v}_Q}(\mathcal{K})^*. \tag{2.33}$$

In other words, $^* : \mathcal{F}^1 \otimes \mathcal{F}^1 \to \mathcal{F}^1 \otimes \mathcal{F}^1$ is a $\mathcal{V}^1$-Lie module automorphism. In particular, the Lie derivative preserves self- and skew-adjointness.

PROOF. Formula (1.21). □

2.7 Functional 2-Vectors

2.7.1 Corollary (Lie derivative of functional 2-vectors)
The Lie derivative of an operator $\mathcal{S} \in \mathcal{V}^1 \otimes \mathcal{V}^1$, *i.e. an operator* $\mathcal{S} : \mathcal{F}^1 \to \mathcal{V}^1$, *especially a functional 2-vector* $\mathcal{S} \in \mathcal{V}^2$, *with respect to the evolutionary vector field* $\mathbf{v}_Q$ *is given by*

$$\mathcal{L}_{\mathbf{v}_Q}\mathcal{S} = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{S}) - \mathsf{D}_Q\mathcal{S} - \mathcal{S}\mathsf{D}_Q^*. \tag{2.34}$$

2.7.2 Proposition
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q}$ *on* $\mathcal{V}^1 \otimes \mathcal{V}^1$ *is a Lie algebra homomorphism, i.e.*

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}],$$

turning $\mathcal{V}^1 \otimes \mathcal{V}^1$ *into a module for the Lie algebra* $\mathcal{V}^1$.

2.7.3 Corollary (Naturality of the adjoint operation)
The Lie derivative commutes with taking adjoints, i.e.

$$\mathcal{L}_{\mathbf{v}_Q}(\mathcal{S}^*) = \mathcal{L}_{\mathbf{v}_Q}(\mathcal{S})^*. \tag{2.35}$$

In other words, $^* : \mathcal{V}^1 \otimes \mathcal{V}^1 \to \mathcal{V}^1 \otimes \mathcal{V}^1$ *is a* $\mathcal{V}^1$*-Lie module automorphism. In particular, the Lie derivative preserves self- and skew-adjointness.*

Proof. Formula (1.21). □

2.8 Functional (1, 1)-Tensors

The delicate point about $(1,1)$-tensors is the fact, that one can view them as differential operators from $\mathcal{V}^1 \to \mathcal{V}^1$, but also as differential operators from $\mathcal{F}^1 \to \mathcal{F}^1$. Taking adjoints is a *natural* isomorphism. We denote the first space by $\mathcal{F}^1 \otimes \mathcal{V}^1$ and the second by $\mathcal{V}^1 \otimes \mathcal{F}^1$.

2.8.1 Corollary (Lie derivative on $\mathcal{F}^1 \otimes \mathcal{V}^1$)
The Lie derivative of an operator $\mathcal{R} \in \mathcal{F}^1 \otimes \mathcal{V}^1$, *i.e. an operator* $\mathcal{R} : \mathcal{V}^1 \to \mathcal{V}^1$, *with respect to the evolutionary vector field* $\mathbf{v}_Q$ *is given by*[10]

$$\mathcal{L}_{\mathbf{v}_Q}\mathcal{R} = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{R}) - \mathsf{D}_Q\mathcal{R} + \mathcal{R}\mathsf{D}_Q. \tag{2.36}$$

2.8.2 Remark (Lie derivative on $\mathcal{V}^1 \otimes \mathcal{F}^1$)
If we view $\mathcal{R}$ as an operator $\mathcal{R} : \mathcal{F}^1 \to \mathcal{F}^1$, then (2.36) must be replaced by

$$\mathcal{L}_{\mathbf{v}_Q}\mathcal{R} = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{R}) + \mathsf{D}_Q^*\mathcal{R} - \mathcal{R}\mathsf{D}_Q^*. \tag{2.37}$$

[10] Cf. Remark 4.3.1.

2.8.3 Proposition
The Lie derivative $\mathcal{L} : \mathbf{v}_Q \mapsto \mathcal{L}_{\mathbf{v}_Q}$ *on* $\mathcal{F}^1 \otimes \mathcal{V}^1$ *(resp.* $\mathcal{V}^1 \otimes \mathcal{F}^1$*) is a Lie algebra homomorphism, i.e.*

$$\mathcal{L}_{[\mathbf{v}_{Q_1},\mathbf{v}_{Q_2}]} = [\mathcal{L}_{\mathbf{v}_{Q_1}}, \mathcal{L}_{\mathbf{v}_{Q_2}}],$$

turning $\mathcal{F}^1 \otimes \mathcal{V}^1$ *(resp.* $\mathcal{V}^1 \otimes \mathcal{F}^1$*) into a module for the Lie algebra* $\mathcal{V}^1$*.*

2.8.4 Corollary (Naturality of the adjoint operation)
The Lie derivative commutes with taking adjoints, i.e.

$$\mathcal{L}_{\mathbf{v}_Q}(\mathcal{R}^*) = \mathcal{L}_{\mathbf{v}_Q}(\mathcal{R})^*. \tag{2.38}$$

In other words, $^* : \mathcal{V}^1 \otimes \mathcal{F}^1 \to \mathcal{F}^1 \otimes \mathcal{V}^1$ *is a* $\mathcal{V}^1$*-Lie module isomorphism.*

PROOF. Formula (1.21). $\square$

Chapter 3

Cartan Formulas and the Euler Complex

3.1 The Interior Product

3.1.1 Definition
For a generalized vector field $\mathbf{v}$ with characteristic $Q \in \mathcal{V}^1$ the *interior product* $\iota_{\mathbf{v}} : \mathcal{F}^s \to \mathcal{F}^{s-1}$ is defined by:

(i) $\iota_{\mathbf{v}} L := 0$ for all $L \in \mathcal{F}^0$.

(ii) $\iota_{\mathbf{v}} \Delta := \Delta \cdot Q$ for all $\Delta \in \mathcal{F}^1$.

(iii) $\iota_{\mathbf{v}} \mathcal{K} := \mathcal{K} Q$ for all $\mathcal{K} \in \mathcal{F}^2$.

(iv) $(\iota_{\mathbf{v}} \mathcal{D})(S_1, \ldots, S_{s-3}) := \mathcal{D}(S_1, \ldots, S_{s-3}, \cdot) Q$ for all $\mathcal{D} \in \mathcal{F}^s$, $(s \geq 3)$.

One can thus restrict the definition to *evolutionary* vector fields.

3.1.2 Remark
This definition coincides up to a factor with the classical one for differential forms. In order for it to be the exact generalization of the classical one, one needs to alter the definition for every $\mathcal{F}^s$, $s \geq 2$. For example, for $\mathcal{K} \in \mathcal{F}^2$ the altered interior product is $\iota_{\mathbf{v}} \mathcal{K} := 2\mathcal{K} Q$. We use the above definition for simplicity.

3.1.3 Lemma
For $Q_1, Q_2 \in \mathcal{V}^1$

(i) $\iota_{\mathbf{v}_{Q_1}} \iota_{\mathbf{v}_{Q_2}} + \iota_{\mathbf{v}_{Q_2}} \iota_{\mathbf{v}_{Q_1}} = 0$.

(ii) $[\mathcal{L}_{\mathbf{v}_{Q_1}}, \iota_{\mathbf{v}_{Q_2}}] = \iota_{[\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}]}$.

Proof. (i) is a direct consequence of the total skew-symmetry of functional forms. (ii) follows from the Leibniz rule (2.26): $[\mathcal{L}_{\mathbf{v}_{Q_1}}, \iota_{\mathbf{v}_{Q_2}}]\mathcal{D} = \mathcal{L}_{\mathbf{v}_{Q_1}} \iota_{\mathbf{v}_{Q_2}} \mathcal{D} - \iota_{\mathbf{v}_{Q_2}} \mathcal{L}_{\mathbf{v}_{Q_1}} \mathcal{D} \overset{(2.1)}{=} \iota_{\mathbf{v}_{Q_2}} \mathcal{L}_{\mathbf{v}_{Q_1}} \mathcal{D} + \iota_{\mathcal{L}_{\mathbf{v}_{Q_1}} \mathbf{v}_{Q_2}} \mathcal{D} - \iota_{\mathbf{v}_{Q_2}} \mathcal{L}_{\mathbf{v}_{Q_1}} \mathcal{D} \overset{(2.16)}{=} \iota_{[\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}]} \mathcal{D}$. □

3.2 The Cartan Formulas

Our aim is to assume the validity of the Cartan formula for the functional spaces $\mathcal{F}^s$, i.e. a formula of the form[1]

$$\mathcal{L}_{\mathbf{v}_Q} = \delta\, \iota_{\mathbf{v}_Q} + \iota_{\mathbf{v}_Q}\delta, \tag{3.1}$$

and *recursively* derive the morphisms $\delta : \mathcal{F}^s \to \mathcal{F}^{s+1}$ of the Euler complex

$$0 \to \mathcal{F}^0 \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\delta} \mathcal{F}^3 \to \cdots$$

This will be explicitly carried out for three steps, i.e. for $\delta : \mathcal{F}^0 \to \mathcal{F}^1$, $\delta : \mathcal{F}^1 \to \mathcal{F}^2$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^3$.

3.2.1 Remark (Local exactness)
Cartan formulas are, among several other applications, used - in an integrated form - to prove *local exactness* of the underlying sequence. More precisely, for the sequence

$$\mathcal{F}^{s-1} \xrightarrow{\delta} \mathcal{F}^s \xrightarrow{\delta} \mathcal{F}^{s+1}$$

the Cartan formula $\mathcal{L}_{\mathbf{v}_Q} = \delta\, \iota_{\mathbf{v}_Q} + \iota_{\mathbf{v}_Q}\delta$ can *locally* be integrated to a *homotopy formula*[2]

$$\mathrm{id} = \delta h + h\delta.$$

Together with $\delta \circ \delta = 0$ it proves the local exactness of the sequence, i.e.

$$\mathrm{Im}(\mathcal{F}^{s-1} \xrightarrow{\delta} \mathcal{F}^s) = \mathrm{Ker}(\mathcal{F}^s \xrightarrow{\delta} \mathcal{F}^{s+1})$$

locally. For details see [Olv], Section 5.4, p. 354.

3.2.2 Theorem (Naturality of δ)
If there exists morphisms $\delta : \mathcal{F}^s \to \mathcal{F}^{s+1}$ for all[3] *$s \geq 0$ satisfying the Cartan formula (3.1), then the morphisms δ are* natural, *i.e.*

$$\mathcal{L}_{\mathbf{v}_Q}(\delta) := [\mathcal{L}_{\mathbf{v}_Q}, \delta] := \mathcal{L}_{\mathbf{v}_Q}\delta - \delta\mathcal{L}_{\mathbf{v}_Q} = 0. \tag{3.2}$$

Proof. Using the simple fact, that the commutator $[\,\cdot,\cdot\,]$ is in each argument a derivation, one verifies for $Q, R \in \mathcal{V}^1$

$$\begin{aligned}
\mathcal{L}_{[\mathbf{v}_Q,\mathbf{v}_R]} &\overset{(2.30)}{=} [\mathcal{L}_{\mathbf{v}_Q}, \mathcal{L}_{\mathbf{v}_R}] \\
&= [\mathcal{L}_{\mathbf{v}_Q}, \delta\, \iota_{\mathbf{v}_R} + \iota_{\mathbf{v}_R}\delta] \\
&= [\mathcal{L}_{\mathbf{v}_Q}, \delta]\, \iota_{\mathbf{v}_R} + \delta\, [\mathcal{L}_{\mathbf{v}_Q}, \iota_{\mathbf{v}_R}] + [\mathcal{L}_{\mathbf{v}_Q}, \iota_{\mathbf{v}_R}]\delta + \iota_{\mathbf{v}_R}[\mathcal{L}_{\mathbf{v}_Q}, \delta] \\
&= \delta\, \iota_{[\mathbf{v}_Q,\mathbf{v}_R]} + \iota_{[\mathbf{v}_Q,\mathbf{v}_R]}\delta + [\mathcal{L}_{\mathbf{v}_Q}, \delta]\, \iota_{\mathbf{v}_R} + \iota_{\mathbf{v}_R}[\mathcal{L}_{\mathbf{v}_Q}, \delta] \\
&\overset{(3.1)}{=} \mathcal{L}_{[\mathbf{v}_Q,\mathbf{v}_R]} + [\mathcal{L}_{\mathbf{v}_Q}, \delta]\, \iota_{\mathbf{v}_R} + \iota_{\mathbf{v}_R}[\mathcal{L}_{\mathbf{v}_Q}, \delta],
\end{aligned}$$

[1] ι_X denotes the interior product with respect to the vector field X: $(\iota_X\omega)(X_2, \ldots, X_k) = \omega(X, X_2, \ldots, X_k)$, for a k-form ω and vector fields X_i.

[2] In the context of the variational bicomplex, it is called the *vertical* homotopy formula.

[3] Actually, only the existence for $s \in \{0, \ldots, N\}$ for some N is needed.

and hence $[\mathcal{L}_{\mathbf{v}_Q}, \delta]\,\iota_{\mathbf{v}_R} + \iota_{\mathbf{v}_R}[\mathcal{L}_{\mathbf{v}_Q}, \delta] = 0$. Since the Cartan formula for $\mathcal{F}^0$ is just $\mathcal{L}_{\mathbf{v}_R} = \iota_{\mathbf{v}_R}\delta$, one deduces from the above calculations, that $[\mathcal{L}_{\mathbf{v}_Q}, \delta] = 0$ for $\delta : \mathcal{F}^0 \to \mathcal{F}^1$, which gets the induction started. By the inductive hypothesis the first term vanishes and therefore the second term must vanish too, giving the inductive step. □

3.2.3 Theorem ($\delta \circ \delta = 0$)
If there exists morphisms $\delta : \mathcal{F}^s \to \mathcal{F}^{s+1}$ for all[4] $s \geq 0$ satisfying the Cartan formula (3.1), then

$$\delta \circ \delta = 0. \tag{3.3}$$

Proof. For all $Q \in \mathcal{V}^1$

$$\begin{aligned} 0 &\overset{(3.2)}{=} [\mathcal{L}_{\mathbf{v}_Q}, \delta] \\ &\overset{(3.1)}{=} [\delta\,\iota_{\mathbf{v}_Q} + \iota_{\mathbf{v}_Q}\delta, \delta] \\ &= \delta\,\iota_{\mathbf{v}_Q}\delta + \iota_{\mathbf{v}_Q}\delta\delta - \delta\delta\iota_{\mathbf{v}_Q} - \delta\,\iota_{\mathbf{v}_Q}\delta \\ &= \iota_{\mathbf{v}_Q}\delta\delta - \delta\delta\iota_{\mathbf{v}_Q}. \end{aligned}$$

Starting the induction at $\mathcal{F}^0$ the second term vanishes and hence the first too, implying $\delta\delta = 0$. By the inductive hypothesis the second term vanishes and hence the first too, giving the inductive step. □

3.3 The Euler Operator

3.3.1 Theorem (Cartan Formula for $\mathcal{F}^0$)
There exists one and only one operator $\mathsf{E} : \mathcal{F}^0 \to \mathcal{F}^1$ satisfying the Cartan formula[5] for $\mathcal{F}^0$, i.e.

$$\mathcal{L}_{\mathbf{v}_Q} L = \mathsf{E}(L) \cdot Q, \tag{3.4}$$

for all functionals[6] $L \in \mathcal{F}^0$ and $Q \in \mathcal{V}^1$.

Proof. First note that $\mathcal{L}_{\mathbf{v}_Q} L = \mathsf{pr}\,\mathbf{v}_Q L \overset{(1.13)}{=} \mathsf{D}_L Q$. Due to the integration by parts formula (1.11) $\mathsf{E}(L) := \mathsf{D}_L^*(1)$ is thus the unique functional 1-form satisfying $\mathsf{D}_L Q = \mathsf{E}(L) \cdot Q \in \mathcal{F}^0$ for all $Q \in \mathcal{V}^1$. $\mathsf{E}(L)$ is merely D_L under the identification of Corollary 1.2.17. The proof will be completed with Lemma 3.3.3. □

[4] Again, only the existence for $s \in \{0, \ldots, N\}$ for some N is needed.

[5] In analogy to the Cartan formula $\mathcal{L}_X f = \iota_X\, df = \langle df, X\rangle$ for a function f on a manifold, where ι_X is the interior product with respect to the vector field X (cf. Section 3.2).

[6] If we view L as a *Lagrangians*, i.e. a horizontal p-form, the Cartan formula then takes the form $\mathcal{L}_{\mathbf{v}_Q} L = \mathsf{Div}(\iota_{\mathbf{v}_Q} L) + \mathsf{E}(L) \cdot Q$, [Olv], Formula (5.135). If we again view L as a functional, then the first term is zero by definition of $\mathcal{F}^0$. I do not want to dwell on the definition of $\iota_{\mathbf{v}_Q} L$ for Lagrangians here.

3.3.2 Definition (Euler operator)
For a Lagrangian $L \in \mathcal{A}$ the operator $\mathsf{E} = \delta : \mathcal{A} \to \mathcal{F}^1$ defined by

$$\mathsf{E}(L) := \mathsf{D}_L^*(1) \tag{3.5}$$

is called the EULER operator. The equation $\mathsf{E}(L) = 0$ is called the EULER-LAGRANGE equation.

The following lemma shows that the Euler operator $\mathsf{E} : \mathcal{A} \to \mathcal{F}^1$ factors over $\mathcal{F}^0 := \mathcal{A}/\mathsf{Div}(\mathcal{A}^p)$, i.e. it can be viewed as an operator

$$\mathsf{E} : \mathcal{F}^0 \to \mathcal{F}^1,$$

which completes the existence part of the proof of the above theorem.

3.3.3 Lemma
The Euler operator $\mathsf{E} : \mathcal{A} \to \mathcal{F}^1$ vanishes on divergences, i.e.

$$\mathsf{E}(\mathsf{Div}\mathcal{A}^p) = 0. \tag{3.6}$$

PROOF. For $P \in \mathcal{A}^p$

$$\begin{aligned}
\mathsf{E}(\mathsf{Div}P) &= \mathsf{D}^*_{\mathsf{Div}P}(1) \\
&= \mathsf{D}^*_{D_i P^i}(1) \\
&\overset{(1.18)}{=} (\underbrace{\mathsf{pr}\,\mathbf{v}.(D_i)}_{=0} P^i + D_i \mathsf{D}_{P^i})^*(1) \\
&= \mathsf{D}^*_{P^i}(-D_i)(1) \\
&= 0.
\end{aligned}$$

□

3.3.4 Corollary
The Cartan formula guarantees the local *exactness of the sequence* $0 \to \mathcal{F}^0 \xrightarrow{\mathsf{E}} \mathcal{F}^1$, *i.e.* $\mathrm{Ker}(\mathsf{E}) = \{0\} \leq \mathcal{F}^0$ *locally, or equivalently the* local *exactness of the sequence* $\mathcal{A}^p \xrightarrow{\mathsf{Div}} \mathcal{F}^0 \xrightarrow{\mathsf{E}} \mathcal{F}^1$, *i.e.* $\mathrm{Ker}(\mathsf{E}) = \mathsf{Div}(\mathcal{A}^p)$ *locally. See Remark 3.2.1 for details.*

3.3.5 Remark
Combining (1.13) and (3.4) we obtain

$$\mathsf{D}_L Q = \mathsf{E}(L) \cdot Q,$$

which, as an identity of functionals, simply asserts that the Fréchet operator $\mathsf{D} : \mathcal{A} \to \mathrm{Hom}(\mathcal{V}^1, \mathcal{A})$ viewed[7] as an operator $\mathsf{D} : \mathcal{F}^0 \to \mathrm{Hom}(\mathcal{V}^1, \mathcal{F}^0)$ is precisely the Euler operator $\mathsf{E} : \mathcal{F}^0 \to \mathcal{F}^1$.

[7]This reflects the fact, that the vertical derivative in the variational bicomplex induces the morphisms of the Euler complex.

3.3.6 Remark (Euler operator)
For $L \in \mathcal{A}$ the Euler operator is given by

$$\mathsf{E}(L) = ((-D)_J \frac{\partial L}{\partial u^1_J}, \ldots, (-D)_J \frac{\partial L}{\partial u^q_J}),$$

where $(-D)_J := (-1)^{|J|} D_J$. The components $\mathsf{E}_\alpha(L)$ are called the *variational derivatives*[8] and often denoted by $\frac{\delta L}{\delta u^\alpha}$.

3.3.7 Example
Let x be a single independent variable and (u, v) the dependent variables. For $L = -\frac{1}{6}u_x^2 + \frac{4}{9}u^3 + \frac{1}{2}v^2$ one computes $\mathsf{E}(L) = (\frac{4}{3}u^2 + \frac{1}{3}u_{xx}, v)$. (See Example 4.3.14).

3.3.8 Example
The Euler-Lagrange expression of the Lagrangian $L = \sqrt{1 + u_x^2 + u_y^2}$ is, up to a sign, the mean curvature

$$\mathsf{E}(L) = -H = -\frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{\sqrt{1 + u_x^2 + u_y^2}^3}.$$

3.3.9 Example
The Camassa-Holm equation

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx} \tag{3.7}$$

is *not* an Euler-Lagrange equation, cf. Example 3.4.5. If one replaces u by v_x the resulting, so called *derived*[9] *potential* Camassa-Holm equation

$$v_{tx} - v_{txxx} = -3v_x v_{xx} + 2v_{xx} v_{xxx} + v_x v_{xxxx} \tag{3.8}$$

becomes Euler-Lagrange with Lagrangian

$$L = vv_x v_{xx} - \frac{1}{2}v_x v_{xx}^2 - \frac{1}{2}v_t v_x - \frac{1}{2}v_{tx} v_{xx}.$$

3.3.10 Example
In all modern gauge field theories the field equations are Euler-Lagrange equations, i.e.

$$\mathsf{E}(L) = 0,$$

for a specific, physically motivated Lagrangian L. As a popular example we can view the 4 space-time coordinates (x^μ) as independent variables, and the

---

[8] In the language of the variational bicomplex: $\delta(L\nu) = \mathsf{E}_\alpha(L)\delta u^\alpha \wedge \nu \in \mathcal{F}^1$ for $L\nu \in \mathcal{F}^0 \equiv \Omega^{p,0}/D(\Omega^{p-1,0})$ and $\nu = dx^1 \wedge \ldots \wedge dx^p$.

[9] It is the x-derivative of the *potential* Camassa-Holm equation $v_t - v_{txx} = -\frac{3}{2}v_x^2 + \frac{1}{2}v_{xx}^2 + v_x v_{xxx}$. See [Olv], Exercise 5.46 & 5.11.

10 independent components of the Lorentzian metric tensor $(g_{\mu\nu})$ as dependent variables. The variational derivatives of the Hilbert-Einstein Lagrangian $L = R\sqrt{|g|}$, where R is the scalar curvature expressed in $g^{\mu\nu}$ and its derivatives, and $g = \det(g_{\mu\nu})$, coincide up to a factor with the components of the Einstein field tensor $G = (G^{\mu\nu})$:

$$\frac{\delta L}{\delta g_{\mu\nu}} = -G^{\mu\nu}\sqrt{|g|},$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ and $(R^{\mu\nu})$ is the Ricci curvature tensor. Thus the Einstein vacuum field equations $G^{\mu\nu} = 0$ arise from a variational principle ([Ste], p. 92). This was first discovered by Hilbert, even before Einstein published his field equations. The other famous example is the Maxwell equations, or more generally, the Yang-Mills equations.

We now prove the following important property of the Euler operator using the second key lemma.

3.3.11 Lemma
The operator $\mathsf{D}_{\mathsf{E}(L)}$ *is self-adjoint*[10]*:*

$$\mathsf{D}^*_{\mathsf{E}(L)} = \mathsf{D}_{\mathsf{E}(L)}. \tag{3.9}$$

Proof. Using the simple fact that $\mathsf{D}_1 = 0$ we obtain

$$\begin{aligned} \mathsf{D}_{\mathsf{E}(L)} &\overset{(3.5)}{=} \mathsf{D}_{\mathsf{D}^*_L(1)} \\ &\overset{(1.18)}{=} \mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_L)1 + \mathsf{D}^*_L\mathsf{D}_1 \\ &\overset{(1.23)}{=} (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_L)1 + \mathsf{D}^*_L\mathsf{D}_1)^* \\ &\overset{(1.18)}{=} \mathsf{D}^*_{\mathsf{E}(L)}. \end{aligned}$$

□

3.3.12 Lemma ([Olv], Formula (5.34))
For $L, P \in \mathcal{A}^r$ *the following* product formula *for the Fréchet derivative holds:*

$$\mathsf{D}_{L\cdot P} = L\mathsf{D}_P + P\mathsf{D}_L, \tag{3.10}$$

where $L \cdot P := \sum_\alpha L_\alpha P_\alpha$.

Proof. The proof follows from the standard product formula. □

3.3.13 Lemma ([Olv], Formula (5.80))
For the Euler operator the following product formula holds:

$$E(L \cdot P) = \mathsf{D}^*_L P + \mathsf{D}^*_P L, \tag{3.11}$$

for all $L, P \in \mathcal{A}^r$.

[10] The proof given here is in the spirit of [Olv], Sec. 5.3, because it avoids the use of differential forms, the tool of the proof in [Olv], Sec. 5.4.

PROOF.

$$\begin{aligned} E(L\cdot P) &\overset{(3.5)}{=} \mathsf{D}^*_{L\cdot P}(1)\\ &\overset{(3.10)}{=} (L\mathsf{D}_P + P\mathsf{D}_L)^*(1)\\ &= (\mathsf{D}^*_P L + \mathsf{D}^*_L P)(1)\\ &= \mathsf{D}^*_P L + \mathsf{D}^*_L P. \end{aligned}$$

□

3.3.14 Proposition (Naturality of E)
For a functional $L \in \mathcal{F}^0$ the following two statements are equivalent:

(i) $\mathsf{E}(L)$ *transforms infinitesimally as a source form.*

(ii) The Lie derivative commutes with E, *i.e. for all $Q \in \mathcal{V}^1$*

$$\mathcal{L}_{\mathbf{v}_Q}\mathsf{E}(L) = \mathsf{E}(\mathcal{L}_{\mathbf{v}_Q}L). \tag{3.12}$$

PROOF. This is a special case of Theorem 3.2.2. The proof of the theorem merely used the Leibniz principle and the Cartan formula. Nevertheless a direct proof is given to demonstrate the calculus developed in Chapter 1. Both directions follow from the following equalities. For an arbitrary $Q \in \mathcal{V}^1$

$$\begin{aligned} \mathsf{E}(\mathcal{L}_{\mathbf{v}_Q}L) &\overset{(3.4)}{=} \mathsf{E}(\mathsf{E}(L)\cdot Q)\\ &\overset{(3.11)}{=} \mathsf{D}^*_{\mathsf{E}(L)}Q + \mathsf{D}^*_Q\mathsf{E}(L)\\ &\overset{(3.9)}{=} \mathsf{D}_{\mathsf{E}(L)}Q + \mathsf{D}^*_Q\mathsf{E}(L)\\ &\overset{(1.13)}{=} \mathsf{pr}\,\mathbf{v}_Q\mathsf{E}(L) + \mathsf{D}^*_Q\mathsf{E}(L), \end{aligned}$$

which by (2.23) completes the proof. □

3.4 The Helmholtz Operator

3.4.1 Theorem (Cartan formula for $\mathcal{F}^1$)
There exists one and only one operator $\mathsf{H} : \mathcal{F}^1 \to \mathcal{F}^2$ satisfying the Cartan formula for $\mathcal{F}^1$, i.e.

$$\mathcal{L}_{\mathbf{v}_Q}\Delta = \mathsf{E}(\Delta\cdot Q) + \mathsf{H}_\Delta(Q), \tag{3.13}$$

for $\Delta \in \mathcal{F}^1$ and $Q \in \mathcal{V}^1$.

PROOF.

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_Q}\Delta - \mathsf{E}(\Delta\cdot Q) &\overset{(2.23),(3.11)}{=} \mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta - \mathsf{D}_Q^*\Delta - \mathsf{D}_\Delta^* Q\\ &\overset{(1.13)}{=} (\mathsf{D}_\Delta - \mathsf{D}_\Delta^*)(Q).\end{aligned}$$

Hence $\mathsf{H}_\Delta(Q) = (\mathsf{D}_\Delta - \mathsf{D}_\Delta^*)(Q)$. Clearly $\mathsf{H}_\Delta \in \mathcal{F}^2$. □

Note $\mathsf{D}_\Delta : \mathcal{V}^1 \to \mathcal{F}^1$ and hence also $\mathsf{D}_\Delta^* : \mathcal{V}^1 \to \mathcal{F}^1$.

3.4.2 Definition (Helmholtz operator)
For $\Delta \in \mathcal{F}^1$ the operator $\mathsf{H} = \delta : \mathcal{F}^1 \to \mathcal{F}^2$ defined by

$$\mathsf{H}_\Delta := \mathsf{D}_\Delta - \mathsf{D}_\Delta^* \tag{3.14}$$

is called the HELMHOLTZ operator[11].

3.4.3 Lemma
The Helmholtz operator satisfies

$$\mathsf{H}_{\mathsf{E}(L)} = 0, \tag{3.15}$$

for all $L \in \mathcal{F}^0$.

PROOF. This is a special case of Theorem 3.2.3. Another, direct proof is provided by Formula (3.9). □

3.4.4 Proposition (Naturality of H)
For a source form $\Delta \in \mathcal{F}^1$ the following two statements are equivalent:

(i) H_Δ transforms infinitesimally as a functional 2-form.

(ii) The Lie derivative commutes with H, i.e. for all $Q \in \mathcal{V}^1$

$$\mathcal{L}_{\mathbf{v}_Q}\mathsf{H}_\Delta = \mathsf{H}_{\mathcal{L}_{\mathbf{v}_Q}\Delta}. \tag{3.16}$$

PROOF. This is a special case of Theorem 3.2.2. Again another, direct proof is given. Both directions follow from the following equalities. For an arbitrary $Q \in \mathcal{V}^1$

$$\begin{aligned}\mathsf{H}_{\mathcal{L}_{\mathbf{v}_Q}\Delta} &\overset{(3.14),(2.23)}{=} \mathsf{D}_{\mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta} - \mathsf{D}^*_{\mathsf{pr}\,\mathbf{v}_Q\Delta + \mathsf{D}_Q^*\Delta}\\ &\overset{(1.24),(1.18)}{=} \mathsf{pr}\,\mathbf{v}_Q(\mathsf{D}_\Delta) + \mathsf{D}_\Delta\mathsf{D}_Q + \mathsf{pr}\,\mathbf{v}.(\mathsf{D}_Q^*)\Delta + \mathsf{D}_Q^*\mathsf{D}_\Delta\end{aligned}$$

[11] In order for the Helmholtz operator to be the exact generalization of the exterior derivative $d : \Omega^1(M) \to \Omega^2(M)$, one must alter the definition: $\mathsf{H}_\Delta := \frac{1}{2}(\mathsf{D}_\Delta - \mathsf{D}_\Delta^*)$. The reason that we did not obtain the factor $\frac{1}{2}$ lies in the definition of interior product. Cf. Remark 3.1.2.

$$
\begin{aligned}
& -\mathsf{pr}\,\mathbf{v}_Q(\mathsf{D}_\Delta)^* - \mathsf{D}_Q^*\mathsf{D}_\Delta^* - (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_Q^*)\Delta)^* - \mathsf{D}_\Delta^*\mathsf{D}_Q \\
\overset{(1.21)}{=} \quad & \mathsf{pr}\,\mathbf{v}_Q(\mathsf{D}_\Delta - \mathsf{D}_\Delta^*) + \mathsf{D}_Q^*(\mathsf{D}_\Delta - \mathsf{D}_\Delta^*) + (\mathsf{D}_\Delta - \mathsf{D}_\Delta^*)\mathsf{D}_Q \\
& + \underbrace{(\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_Q^*)\Delta - (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_Q^*)\Delta)^*)}_{\overset{(1.23)}{=}0} \\
\overset{(3.14)}{=} \quad & \mathsf{pr}\,\mathbf{v}_Q(\mathsf{H}_\Delta) + \mathsf{D}_Q^*\mathsf{H}_\Delta + \mathsf{H}_\Delta\mathsf{D}_Q,
\end{aligned}
$$

which by (2.31) completes the proof. □

3.4.5 Example
The Camassa-Holm equation (3.7) is not *variational*, i.e. not of the form $\mathsf{E}(L)$. This is shown by verifying $\mathsf{H}_\Delta \neq 0$ for $\Delta = u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}$. Indeed,

$$\mathsf{H}_\Delta = (-2D_{txx} - 2uD_{xxx} - 3u_x D_{xx} + 2D_t + (6u - 3u_{xx})D_x + (3u_x - u_{xxx})) \neq 0.$$

In contrast, equation (3.8) is variational. Indeed $\mathsf{H}_\Delta = 0$ for $\Delta = v_{tx} - v_{txxx} + 3v_x v_{xx} - 2v_{xx}v_{xxx} - v_x v_{xxxx}$.

3.4.6 Corollary (Noether theorem)
Let Δ be an Euler-Lagrange form, i.e. $\Delta = \mathsf{E}(L)$ for some Lagrangian L. Then $\mathbf{v}_Q$ is a distinguished symmetry of Δ, if and only if Q is a characteristic of a local conservation law[12]*, i.e. there exists locally a current $P \in \mathcal{A}^p$, such that $\mathsf{E}(L) \cdot Q = \mathsf{Div} P$.*

Proof. For $\Delta = \mathsf{E}(L)$ we have $\mathsf{H}_{\mathsf{E}(L)} = 0$ by Lemma 3.4.3. So the Cartan formula (3.13) implies that $\mathfrak{L}_{\mathbf{v}_Q}\mathsf{E}(L) = 0$ is equivalent to $\mathsf{E}(\mathsf{E}(L) \cdot Q) = 0$, which is by Corollary 3.3.4 equivalent to the local existence of a current P, such that $\mathsf{E}(L) \cdot Q = \mathsf{Div} P$.

□

3.4.7 Example
The following vector fields are Bessel-Hagen symmetries of the derived potential Camassa-Holm equation (3.8), interpreted as a source equation:

(i) $\mathbf{v}_1 := f(t, 2v_t - 2v_{txx} - v_{xx}^2 - 2v_x v_{xxx} + 3v_x^2)\partial_v$.

(ii) $\mathbf{v}_2 := v_t \partial_v$.

(iii) $\mathbf{v}_3 := v_x \partial_v$.

(iv) $\mathbf{v}_4 := (-\frac{1}{2}v_{xx}^2 v_x + v_{ttx} - v_x v_{txx} + v_{tx}v_{xx} - 2v_x^2 v_{xxx} + \frac{5}{2}v_x^3)\partial_v$.

(v) $\mathbf{v}_5 := (2v_x^3 v_{xxx} + 3v_t^2 + 3v_{tx}^2 - 3v_t v_{xx}^2 - 6v_t v_{txx} + 9v_t v_x^2 - 6v_t v_x v_{xxx} + 2v_{ttt} - 2v_x^4)\partial_v$.

[12] Cf. Corollary 2.4.9.

For the special case $\mathbf{v}_6 = F(t)\partial_v$ the conserved current is $P_6 = (0, \frac{1}{2}F(t)(2v_t - 2v_{txx} - v_{xx}^2 - 2v_x v_{xxx} + 3v_x^2))$. Similarly for $\mathbf{v}_7 = G(t)(2v_t - 2v_{txx} - v_{xx}^2 - 2v_x v_{xxx} + 3v_x^2)\partial_v$ one obtains $P_7 = (0, \frac{1}{4}(2v_t - 2v_{txx} - v_{xx}^2 - 2v_x v_{xxx} + 3v_x^2)^2)$, and for $\mathbf{v}_3$ one obtains $P_3 = (\frac{1}{2}v_x^2 + \frac{1}{2}v_{xx}^2, v_x(v_x^2 - v_x v_{xxx} - v_{txx}))$. The expressions for P_4 and P_5 are too long to be reproduced.

3.5 The Takens Operator

As a preparation for the next operator we prove the following formula, which generalizes Formula (1.23).

3.5.1 Lemma
The following identity holds for a general $\mathcal{K} : \mathcal{V}^1 \to \mathcal{F}^1$

$$(\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q)^* R = (\mathsf{pr}\,\mathbf{v}.(\mathcal{K}^*)R)^* Q. \tag{3.17}$$

PROOF. For an arbitrary characteristic S

$$\begin{aligned}
&\mathsf{E}(S \cdot ((\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q)^* R - (\mathsf{pr}\,\mathbf{v}.(\mathcal{K}^*)R)^* Q)) \\
&\quad = \mathsf{E}(\mathsf{pr}\,\mathbf{v}_S(\mathcal{K})Q \cdot R - \mathsf{pr}\,\mathbf{v}_S(\mathcal{K}^*)R \cdot Q) \\
&\quad = \mathsf{E}(\mathsf{pr}\,\mathbf{v}_S(\mathcal{K})Q \cdot R - R \cdot \mathsf{pr}\,\mathbf{v}_S(\mathcal{K})Q) \\
&\quad = 0.
\end{aligned}$$

□

3.5.2 Theorem (Cartan formula for $\mathcal{F}^2$)
There exists one and only one operator $\mathsf{T} : \mathcal{F}^2 \to \mathcal{F}^3$ satisfying the Cartan formula for $\mathcal{F}^2$, i.e.

$$\mathcal{L}_{\mathbf{v}_Q}\mathcal{K} = \mathsf{H}_{\mathcal{K}Q} + \mathsf{T}_{\mathcal{K}}(Q), \tag{3.18}$$

for all $\mathcal{K} \in \mathcal{F}^2$ and $Q \in \mathcal{V}^1$.

PROOF.

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}_Q}\mathcal{K} - \mathsf{H}_{\mathcal{K}Q} &\overset{(2.31),(3.14)}{=} \mathsf{pr}\,\mathbf{v}_Q(\mathcal{K}) + \mathsf{D}_Q^*\mathcal{K} + \mathcal{K}\mathsf{D}_Q \\
&\qquad -\mathsf{D}_{\mathcal{K}Q} + \mathsf{D}_{\mathcal{K}Q}^* \\
&\overset{(1.18)}{=} \mathsf{pr}\,\mathbf{v}_Q(\mathcal{K}) + \mathsf{D}_Q^*\mathcal{K} + \mathcal{K}\mathsf{D}_Q \\
&\qquad -\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q - \mathcal{K}\mathsf{D}_Q + (\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q)^* + \mathsf{D}_Q^*\mathcal{K}^* \\
&\overset{\mathcal{K}^*=-\mathcal{K}}{=} \mathsf{pr}\,\mathbf{v}_Q(\mathcal{K}) - \mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q + (\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q)^*.
\end{aligned}$$

Hence $\mathsf{T}_{\mathcal{K}}(Q) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{K}) - \mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q + (\mathsf{pr}\,\mathbf{v}.(\mathcal{K})Q)^*$. That $\mathsf{T}_{\mathcal{K}}(Q)$ is skew-adjoint follows immediately from $\mathcal{K}^* = -\mathcal{K}$ and that $\mathsf{T}_{\mathcal{K}}(Q)R = -\mathsf{T}_{\mathcal{K}}(R)Q$ one uses (3.17). Hence $\mathsf{T}_{\mathcal{K}} \in \mathcal{F}^3$. □

3.5.3 Definition (Takens operator)
For $\mathcal{K} \in \mathcal{F}^2$ and $S \in \mathcal{V}^1$ the operator $\mathsf{T} = \delta : \mathcal{F}^2 \to \mathcal{F}^3$ defined by

$$\mathsf{T}_{\mathcal{K}}(S) := \mathsf{pr}\,\mathbf{v}_S(\mathcal{K}) - \mathsf{pr}\,\mathbf{v}.(\mathcal{K})S + (\mathsf{pr}\,\mathbf{v}.(\mathcal{K})S)^* \tag{3.19}$$

is called the TAKENS[13] operator.

3.5.4 Lemma
The Takens operator satisfies

$$\mathsf{T}_{\mathsf{H}_\Delta} = 0, \tag{3.20}$$

for all $\Delta \in \mathcal{F}^1$.

PROOF. Again, this is a special case of Theorem 3.2.3. The following equalities provide another, direct proof. For all $S \in \mathcal{V}^1$

$$\begin{aligned}
\mathsf{T}_{\mathsf{H}_\Delta}(S) &= \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_\Delta - \mathsf{D}^*_\Delta) \\
&\quad -\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta - \mathsf{D}^*_\Delta)S + (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta - \mathsf{D}^*_\Delta)S)^* \\
&= \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_\Delta) - \mathsf{pr}\,\mathbf{v}_S(\mathsf{D}^*_\Delta) \\
&\quad -\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta)S + \mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_\Delta)S \\
&\quad +(\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta)S)^* - (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_\Delta)S)^* \\
&= (\mathsf{pr}\,\mathbf{v}_S(\mathsf{D}_\Delta) - \mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta)S) \\
&\quad -(\mathsf{pr}\,\mathbf{v}_S(\mathsf{D}^*_\Delta) - (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}_\Delta)S)^*) \\
&\quad +(\mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_\Delta)S - (\mathsf{pr}\,\mathbf{v}.(\mathsf{D}^*_\Delta)S)^*) \\
&\overset{(1.22),(1.21),(1.23)}{=} 0.
\end{aligned}$$

□

3.5.5 Proposition (Naturality of T)
For a functional 2*-form* $\mathcal{K} \in \mathcal{F}^2$ *the following two statements are equivalent:*

(i) $\mathsf{T}_{\mathcal{K}}$ *transforms infinitesimally as a functional* 3*-form.*

(ii) The Lie derivative commutes with T*, i.e. for all* $Q \in \mathcal{V}^1$

$$\mathcal{L}_{\mathbf{v}_Q}\mathsf{T}_{\mathcal{K}} = \mathsf{T}_{\mathcal{L}_{\mathbf{v}_Q}\mathcal{K}}. \tag{3.21}$$

[13] In [Tak], a work generalizing Noether's theorem (Corollary 3.4.6), Formula (3.20) $\mathsf{T}_{\mathsf{H}_\Delta} = 0$ is implicitly used. See also [AP], Proposition 3.1 and the discussion following Proposition 3.3. This is why I call T the TAKENS operator.

PROOF. Again, this is a special case of Theorem 3.2.2. □

We thus succeeded to *explicitly* construct the first three morphisms of the celebrated Euler sequence

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\mathsf{E}} \mathcal{F}^1 \xrightarrow{\mathsf{H}} \mathcal{F}^2 \xrightarrow{\mathsf{T}} \mathcal{F}^3 \longrightarrow \cdots$$

In the following two examples we make use of the local exactness of the Euler complex at $\mathcal{F}^2$.

3.5.6 Example
For a given $\mathcal{H} \in \mathcal{F}^2$,

$$\mathcal{H} = \begin{pmatrix} 2v^2yD_x + 2vyv_x & 2u_xvy \\ -2u_xvy & 2xD_{xxy} + 2D_{xy} \end{pmatrix},$$

we want to decide whether there exists a $\Delta \in \mathcal{F}^1$ with $\mathcal{H} = \mathsf{H}_\Delta$. We verify for an arbitrary $T \in \mathcal{V}^1$

$$\mathsf{T}_{\mathcal{H}}(T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and indeed $\mathcal{H} = \mathsf{H}_\Delta$ for $\Delta = (u_xv^2y, xv_{xxy})$.

3.5.7 Example
Modifying the above example slightly and asking the same question for $\mathcal{K} \in \mathcal{F}^2$

$$\mathcal{K} = \mathcal{H} + \begin{pmatrix} 0 & u_xv \\ -u_xv & 0 \end{pmatrix}$$

we obtain the negative answer

$$\mathsf{T}_{\mathcal{K}}(T) = \begin{pmatrix} -2vT^vD_x - v_xT^v - vT^v_x & vT^uD_x + 2vT^u_x + v_xT^u \\ vT^uD_x - vT^u_x & 0 \end{pmatrix} \neq 0,$$

where $T^\alpha_J := D_JT^\alpha$.

3.5.8 Remark
The vanishing of the Takens operator applied to a functional 2-form precisely states its closedness. Thus the Takens operator enables us to formulate an analogue of the symplectic condition. For this we have to enlarge our functional spaces to include total integro-differential operators. Still all the formulas concerning the Euler complex remain valid. The linearity of the symplectic condition is its major advantage compared with the nonlinear Hamiltonian condition (4.7). This will be investigated in a later work.

Chapter 4

Hamiltonian Systems

4.1 Nijenhuis-Schouten Bracket

4.1.1 Definition (Nijenhuis-Schouten bracket)
For $\mathcal{D}, \mathcal{E} \in \mathcal{V}^2$ the *Nijenhuis-Schouten bracket* $[\mathcal{D}, \mathcal{E}] : \mathcal{F}^1 \times \mathcal{F}^1 \times \mathcal{F}^1 \to \mathcal{F}^0$ is defined as follows:

$$[\mathcal{D}, \mathcal{E}](\Delta_1, \Delta_2, \Delta_3) \quad := \quad \mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2 \cdot \mathcal{E}\Delta_3 + \mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2 \cdot \mathcal{D}\Delta_3 + (\text{cycle}), \quad (4.1)$$

where the word (cycle) means summation over all cyclic permutations of the indices $1, 2, 3$. $\mathcal{D}$ and $\mathcal{E}$ are viewed as differential operators from $\mathcal{F}^1$ into $\mathcal{V}^1$.

This definition is a generalization of the classical Nijenhuis-Schouten bracket from differential geometry, which is one of its advantages. It appears in [GDo2], Formula (3.3). Nevertheless there are two major *drawbacks* of this definition. The first one is that the right hand side is a functional, thus it has no normal form. This means that checking the vanishing of the bracket or extracting conditions for its vanishing is not a direct procedure. The second one is that one needs more than total differentials of the Δ_i's, meaning that we cannot compute with general Δ_i's, complicating the check of vanishing of the bracket. Besides, from this definition we do not see that the bracket of two 2-vectors is a $(3, 0)$-tensor, *even* a 3-vector. In the following we want to make use of the freedom of adding divergences to circumvent these drawbacks. The following formula cures both drawbacks.

4.1.2 Proposition (Nijenhuis-Schouten bracket)
The following bracket $[\cdot, \cdot] : \mathcal{V}^2 \times \mathcal{V}^2 \to \mathcal{V}^3$ is an equivalent definition of the Nijenhuis-Schouten bracket

$$\begin{aligned} [\mathcal{D}, \mathcal{E}](\Delta) \quad = \quad & \mathsf{pr}\, \mathbf{v}_{\mathcal{D}\Delta}(\mathcal{E}) - \mathsf{pr}\, \mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta + (\mathsf{pr}\, \mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta)^* + \\ & \mathsf{pr}\, \mathbf{v}_{\mathcal{E}\Delta}(\mathcal{D}) - \mathsf{pr}\, \mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta + (\mathsf{pr}\, \mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta)^*, \end{aligned} \quad (4.2)$$

with $\Delta \in \mathcal{F}^1$ arbitrary.

PROOF. For arbitrary $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{F}^1$

$$
\begin{aligned}
&\mathsf{E}([\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3))\\
&= \mathsf{E}(\mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+(\text{cycle})\\
&\overset{(2.24)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D}\Delta_1))+\\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E}\Delta_1))+\\
&\quad (\text{cycle})\\
&\overset{(1.17)}{=} \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)+\mathsf{E}(\Delta_2\cdot\mathcal{D}\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1)+\\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)+\mathsf{E}(\Delta_2\cdot\mathcal{E}\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1)+\\
&\quad (\text{cycle})\\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)-\mathsf{E}(\mathcal{D}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1)+\\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)-\mathsf{E}(\mathcal{E}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1)+\\
&\quad (\text{cycle})\\
&= \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)-\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1\cdot\mathcal{D}\Delta_2)+\\
&\quad \mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)-\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1\cdot\mathcal{E}\Delta_2)+\\
&\quad (\text{cycle})\\
&\overset{(\text{cycle})}{=} \mathsf{E}(\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2+\Delta_1\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3+\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1\\
&\quad +\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2+\Delta_1\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3+\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)\\
&= \mathsf{E}(\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2-\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_1\cdot\Delta_3+(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^*\Delta_2\cdot\Delta_3\\
&\quad +\Delta_3\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2-\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_1\cdot\Delta_3+(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*\Delta_2\cdot\Delta_3)\\
&= \mathsf{E}(\Delta_3\cdot(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})-\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1+(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^*\\
&\qquad +\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})-\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1+(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*)\Delta_2),
\end{aligned}
$$

proving the desired formula for $\Delta = \Delta_1$. The skew-adjointness of $[\mathcal{D},\mathcal{E}](\Delta)$ follows immediately from the skew-adjointness of $\mathcal{D}$ and $\mathcal{E}$. For $[\mathcal{D},\mathcal{E}](\Delta)\Sigma = -[\mathcal{D},\mathcal{E}](\Sigma)\Delta$ one further needs to notice that $\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta = (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{E})\Delta)\mathcal{D}$ and (3.17) for functional bi-vectors, i.e. skew-adjoint operators $\mathcal{K} : \mathcal{F}^1 \to \mathcal{V}^1$. □

4.1.3 Remark

The above definition is part of the general definition of the Nijenhuis-Schouten bracket

$$[\cdot,\cdot] : \mathcal{V}^r \times \mathcal{V}^s \to \mathcal{V}^{r+s-1},$$

which turns $\bigoplus_{s=0}^{\infty} \mathcal{V}^s$ into a Lie superalgebra ($\mathcal{V}^0 := \mathcal{F}^0$), with $\deg(\mathcal{V}^s) = s-1$. For $L, P \in \mathcal{F}^0$, $Q \in \mathcal{V}^1$ and $\mathcal{S} \in \mathcal{V}^s$ arbitrary, one defines[1] $[L,P] = [P,L] := 0$, $[Q,L] = -[L,Q] := \mathcal{L}_{\mathbf{v}_Q}L = \mathsf{E}(L)\cdot Q = \iota_{\mathsf{E}(L)}Q$, $[\mathcal{S},L] = (-1)^s[L,\mathcal{S}] := \iota_{\mathsf{E}(L)}\mathcal{S}$ and $[Q,\mathcal{S}] = -[\mathcal{S},Q] := \mathcal{L}_{\mathbf{v}_Q}\mathcal{S}$. Thus the Nijenhuis-Schouten bracket is a generalization of the *Lie derivative* for functional multi-vectors.

[1]For $\Sigma \in \mathcal{F}^1$ the interior product $\iota_\Sigma : \mathcal{V}^s \to \mathcal{V}^{s-1}$ is defined completely analogues to the dual one (cf. Definition 3.1.1).

4.1.4 Remark
The right hand side of the formula

$$\begin{aligned}
&[\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3) \\
&= \Delta_3\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 + \Delta_1\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3 + \Delta_2\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1 \\
&\quad +\Delta_3\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 + \Delta_1\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3 + \Delta_2\cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1,
\end{aligned} \tag{4.3}$$

which is part of the proof, appears as Formula (7.30) in [Olv]. This formula is an identity of functionals. This definition still has the first drawback, that trivial functionals do not in general vanish identically, but only up to local divergence. The second drawback is eliminated and one can see the $(3,0)$-tensoriality of the expression. That this expression is in fact a 3-vector is, due to the first drawback, *not* completely easy to see.

4.1.5 Definition (Poisson bracket)
Let $\mathcal{D}$ be a functional $(2,0)$-tensor. The *Poisson bracket* of two functionals L, P is defined by

$$\{L,P\} = \mathsf{E}(L)\cdot\mathcal{D}\mathsf{E}(P), \tag{4.4}$$

which is again a functional.

4.2 Hamiltonian Structure

4.2.1 Definition (Hamiltonian structure)
A functional $(2,0)$-tensor $\mathcal{D}$ is called *Hamiltonian* if its Poisson bracket (4.4) is skew-symmetric

$$\{L,P\} = -\{P,L\}, \tag{4.5}$$

and satisfies the Jacobi identity

$$\{\{L,P\},R\} + \{\{P,R\},L\} + \{\{R,L\},P\} = 0, \tag{4.6}$$

for all functionals L, P, R. These are identities of functionals.

4.2.2 Proposition
A functional $(2,0)$-tensor $\mathcal{D}$ is Hamiltonian, if and only if $\mathcal{D}$ is a 2-vector satisfying

$$[\mathcal{D},\mathcal{D}] = 0. \tag{4.7}$$

Proof. First we note that if we replace $\mathcal{E}$ by $\mathcal{D}$ in the right hand side of (4.3), then, up to a factor, we obtain (7.11) in [Olv]. The rest is done by [Olv] Propositions 7.3, 7.4. ☐

4.2.3 Definition (Hamiltonian equations)
Let $K \in \mathcal{V}^1$ and $u_t = K$ be a system of evolution[2] equations. We say that the evolution equation is *Hamiltonian*, if there exists a Hamiltonian structure $\mathcal{D}$ and a functional H, such that

$$K = \mathcal{D}\mathsf{E}(H). \tag{4.8}$$

Formula (2.34), prescribing the Lie derivative on functional 2-vectors, gives the following structural interpretation of the powerful Lemma 7.26 in [Olv].

4.2.4 Theorem (Criterion for Hamiltonian structures)
Let $u_t = K = \mathcal{D}\mathsf{E}(H)$ be a Hamiltonian system of evolution equations. Then $\mathcal{D}$ is invariant under the flow of $\mathbf{v}_K$, *i.e.*

$$\mathcal{L}_{\mathbf{v}_K}\mathcal{D} = 0. \tag{4.9}$$

$\mathcal{L}_{\mathbf{v}_K}$ is the Lie derivative of functional 2-vectors.

Proof. $K = \mathcal{D}\Delta$ with $\Delta = \mathsf{E}(H)$:

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}_K}\mathcal{D} &\overset{(2.34)}{=} \mathsf{pr}\,\mathbf{v}_K(\mathcal{D}) - \mathsf{D}_K\mathcal{D} - \mathcal{D}\mathsf{D}_K^*\\
&\overset{(1.18)}{=} \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta}(\mathcal{D}) - (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta + \mathcal{D}\mathsf{D}_\Delta)\mathcal{D} - \mathcal{D}(\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta + \mathcal{D}\mathsf{D}_\Delta)^*\\
&\overset{\mathcal{D}^*=-\mathcal{D}}{=} \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta}(\mathcal{D}) - (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta)\mathcal{D} + \mathcal{D}^*(\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta)^*\\
&\qquad -\mathcal{D}\mathsf{D}_\Delta\mathcal{D} + \mathcal{D}\mathsf{D}_\Delta^*\mathcal{D}\\
&\overset{(3.9)}{=} \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta}(\mathcal{D}) - (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta)\mathcal{D} + ((\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{D})\Delta)\mathcal{D})^*\\
&= \frac{1}{2}[\mathcal{D},\mathcal{D}](\Delta)\\
&\overset{(4.7)}{=} 0.
\end{aligned}$$

□

4.2.5 Remark
The usefulness of this theorem lies in the fact, that we can use it as a criterion to determine all Hamiltonian structures, up to a given order, of a given system of evolution equations. This is done for the KdV and the Boussinesq equation.

4.3 Recursion Operators

For what follows let us denote

$$\mathcal{R}\mathbf{v}_Q := \mathbf{v}_{\mathcal{R}Q},$$

and speak about the action of functional $(1,1)$-tensors on evolutionary vector fields.

[2] $u_t = K$ is a short form of $u_t^\alpha = K^\alpha$, $\alpha = 1, \ldots, q$.

4.3.1 Remark
By the above convention, the Leibniz rule (2.26) coincides with [Olv], Formula (5.41):

$$[\mathbf{v}_P, \mathcal{R}\mathbf{v}_Q] = \mathcal{L}_{\mathbf{v}_P}(\mathcal{R})\mathbf{v}_Q + \mathcal{R}[\mathbf{v}_P, \mathbf{v}_Q],$$

which he uses to define the Lie derivative[3] for functional $(1,1)$-tensors. In this thesis the general Leibniz rule (2.1) is declared to be the major property defining the Lie derivate. This enabled us to recursively determine the Lie derivative for *all* functional tensor spaces.

4.3.2 Definition (Recursion operator)
For a system of evolution equations, an operator $\mathcal{R} : \mathcal{V}^1 \to \mathcal{V}^1$ is called *recursion operator* if it maps evolutionary symmetries to evolutionary symmetries.

4.3.3 Corollary (Criterion for recursion operators)
Let $P \in \mathcal{V}^1$ and $u_t = P$ an evolution equation. If the functional $(1,1)$-tensor $\mathcal{R}$ is invariant under the flow of $\mathbf{v}_P$, i.e.

$$\mathcal{L}_{\mathbf{v}_P}(\mathcal{R}) = 0, \tag{4.10}$$

then $\mathcal{R}$ is a recursion operator of the evolution equation.

Proof. If Q is a characteristic of a symmetry then by Lemma 2.3.12 $\mathcal{L}_{\mathbf{v}_P} Q = 0$. Using (4.10) and (2.26) we deduce $\mathcal{L}_{\mathbf{v}_P}(\mathcal{R}Q) = 0$. Again by the Lemma 2.3.12 $\mathcal{R}Q$ is a characteristic of a symmetry. □

4.3.4 Lemma
For a functional $(2,0)$-tensor[4] $\mathcal{E} : \mathcal{F}^1 \to \mathcal{V}^1$ and a functional $(0,2)$-tensor[5] $\mathcal{H} : \mathcal{V}^1 \to \mathcal{F}^1$, the following two statements are equivalent:

(i) $\mathcal{E} \cdot \mathcal{H} : \mathcal{V}^1 \to \mathcal{V}^1$ transforms as a functional $(1,1)$-tensor, i.e. according to (2.36).

(ii) $\mathcal{L}_{\mathbf{v}_Q}$ satisfies the Leibniz rule

$$\mathcal{L}_{\mathbf{v}_Q}(\mathcal{E} \cdot \mathcal{H}) = \mathcal{L}_{\mathbf{v}_Q}\mathcal{E} \cdot \mathcal{H} + \mathcal{E} \cdot \mathcal{L}_{\mathbf{v}_Q}\mathcal{H}. \tag{4.11}$$

Proof. The proof is done by the following equalities:

$$\begin{aligned}
&\mathcal{L}_{\mathbf{v}_Q}(\mathcal{E}) \cdot \mathcal{H} + \mathcal{E} \cdot (\mathcal{L}_{\mathbf{v}_Q}\mathcal{H}) \\
&\overset{(2.31),(2.34)}{=} (\mathsf{pr}\,\mathbf{v}_Q(\mathcal{E}) - \mathsf{D}_Q\mathcal{E} - \mathcal{E}\mathsf{D}_Q^*) \cdot \mathcal{H} + \mathcal{E} \cdot (\mathsf{pr}\,\mathbf{v}_Q(\mathcal{H}) + \mathsf{D}_Q^*\mathcal{H} + \mathcal{H}\mathsf{D}_Q) \\
&\overset{(1.19)}{=} \mathsf{pr}\,\mathbf{v}_Q(\mathcal{E} \cdot \mathcal{H}) + (\mathcal{E} \cdot \mathcal{H})\mathsf{D}_Q - \mathsf{D}_Q(\mathcal{E} \cdot \mathcal{H}),
\end{aligned}$$

which by (2.36) completes the proof. □

[3] [Olv] denotes the Lie derivative $\mathcal{L}_{\mathbf{v}_P}(\mathcal{R})$ by $\mathbf{v}_P[[\mathcal{R}]]$.
[4] Specially a functional 2-vector.
[5] Specially a functional 2-form.

4.3.5 Remark
For $\mathcal{H}\cdot\mathcal{E}:\mathcal{F}^1\to\mathcal{F}^1$ (2.36) must be replaced by (2.37) in the previous lemma.

4.3.6 Lemma
If $\mathcal{D}$ (resp. $\mathcal{K}$) is nondegenerate functional $(2,0)$-tensor (resp. $(0,2)$-tensor) then $\mathcal{D}^{-1}$ (resp. $\mathcal{K}^{-1}$) transforms infinitesimally as a functional $(0,2)$-tensor (resp. $(2,0)$-tensor). Taking inverse preserves self- and skew-adjointness.

PROOF. From $0\overset{(2.36)}{=}\mathcal{L}_{\mathbf{v}_Q}(\mathcal{I}d)=\mathcal{L}_{\mathbf{v}_Q}(\mathcal{D}\cdot\mathcal{D}^{-1})\overset{(4.11)}{=}\mathcal{L}_{\mathbf{v}_Q}\mathcal{D}\cdot\mathcal{D}^{-1}+\mathcal{D}\cdot\mathcal{L}_{\mathbf{v}_Q}(\mathcal{D}^{-1})$ we deduce that

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_Q}(\mathcal{D}^{-1}) &= -\mathcal{D}^{-1}\mathcal{L}_{\mathbf{v}_Q}(\mathcal{D})\mathcal{D}^{-1}\\ &\overset{(2.34)}{=} -\mathcal{D}^{-1}(\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})-\mathsf{D}_Q\mathcal{D}-\mathcal{D}\mathsf{D}_Q^*)\mathcal{D}^{-1}\\ &= -\mathcal{D}^{-1}\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})\mathcal{D}^{-1}+\mathcal{D}^{-1}\mathsf{D}_Q+\mathsf{D}_Q^*\mathcal{D}^{-1}\\ &= \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}^{-1})+\mathcal{D}^{-1}\mathsf{D}_Q+\mathsf{D}_Q^*\mathcal{D}^{-1},\end{aligned}$$

which by (2.31) completes the proof. The rest is proved in a similar fashion. □

4.3.7 Corollary (Construction of recursion operators)
Let $P\in\mathcal{V}^1$ and $u_t=P$ an evolution equation.

(i) If $\mathcal{E}$ is an invariant functional $(2,0)$-tensor (i.e. $\mathcal{L}_{\mathbf{v}_P}\mathcal{E}=0$) and $\mathcal{K}$ an invariant functional $(0,2)$-tensor ($\mathcal{L}_{\mathbf{v}_P}\mathcal{K}=0$), then $\mathcal{R}:=\mathcal{E}\cdot\mathcal{K}$ is a recursion operator of $u_t=P$.

(ii) If $\mathcal{D},\mathcal{E}$ are invariant functional $(2,0)$-tensors (i.e. $\mathcal{L}_{\mathbf{v}_P}\mathcal{D}=\mathcal{L}_{\mathbf{v}_P}\mathcal{E}=0$) and $\mathcal{D}$ nondegenerate, then $\mathcal{R}:=\mathcal{E}\cdot\mathcal{D}^{-1}$ is a recursion operator of $u_t=P$.

PROOF. The proof is done by the above lemmas and (4.10). □

4.3.8 Corollary
Let $u_t=P=\mathcal{E}\mathsf{E}(H_0)=\mathcal{D}\mathsf{E}(H_1)$ be a system with two Hamiltonian structures. If $\mathcal{D}$ is nondegenerate, then

$$\mathcal{R}:=\mathcal{E}\cdot\mathcal{D}^{-1} \tag{4.12}$$

is a recursion operator.

PROOF. This follows from Theorem 4.2.4 and the previous corollary. □

4.3.9 Corollary
Let $u_t=P$ be an evolution equation. If $\mathcal{D}\in\mathcal{V}^1\otimes\mathcal{V}^1$ (resp. $\mathcal{D}\in\mathcal{F}^1\otimes\mathcal{F}^1$) is invariant under the flow of $\mathbf{v}_P$, then the self-adjoint part $\frac{1}{2}(\mathcal{D}+\mathcal{D}^*)$ and the skew-adjoint part $\frac{1}{2}(\mathcal{D}-\mathcal{D}^*)$ are invariant under the flow of $\mathbf{v}_P$.

PROOF. This follows from Formula (2.35) (resp. (2.33)). □

4.3.10 Corollary
Let $u_t = P$ be an evolution equation. If $\mathcal{R}$ is a recursion operator and $\mathcal{D} \in \mathcal{V}^1 \otimes \mathcal{V}^1$ is invariant under the flow of $\mathbf{v}_P$, then $\mathcal{R}\mathcal{D} \in \mathcal{V}^1 \otimes \mathcal{V}^1$ is again invariant under the flow of $\mathbf{v}_P$. The statement remains valid, if $\mathcal{V}^1$ is replaced by $\mathcal{F}^1$ and $\mathcal{R}\mathcal{D}$ by $\mathcal{R}^*\mathcal{D}$.

Proof. This follows from a Leibniz rule analogues to (4.11). □

4.3.11 Definition (Bi-Hamiltonian structure)
Let $u_t = \mathcal{E}\mathsf{E}(H_0) = \mathcal{D}\mathsf{E}(H_1)$ be a system with two Hamiltonian structures. The system is called *bi-Hamiltonian*, if the two structures are *compatible*, i.e. if

$$[\mathcal{D}, \mathcal{E}] = 0.$$

4.3.12 Remark
Bi-Hamiltonian systems with a nondegenerate $\mathcal{D}$ possess a recursion operator $\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$ generating an infinite family of *Hamiltonian symmetries*[6], which by the Hamiltonian version of Noether's theorem gives rise to an infinite family of conservation laws. This is typical for an integrable system. Further details are found in [Olv], Chapter 7.

As mentioned in the introduction, we are now able to explicitly compute Hamiltonian structures and recursion operators of nonlinear completely integrable differential equations. We carry this out for both the KdV and the Boussinesq equation. Because we cannot parametrise the nonlinear space of Hamiltonian 2-vectors, we instead compute the space of functional $(2,0)$-tensors, invariant under the flow, and then determine the Hamiltonian 2-vectors among them. The invariance condition of Theorem 4.2.4 produces a large system of linear PDEs which was generated using the package `jets` and solved with the aid of computer. (See Appendix A)

4.3.13 Example (Korteweg-de Vries equation)
The KdV equation

$$u_t = u_{xxx} + uu_x$$

has a bi-Hamiltonian structure given by

$$\begin{array}{lcl} \mathcal{D} = D_x & , & H_1 = \frac{1}{6}u^3 - \frac{1}{2}u_x^2, \\ \mathcal{E} = D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x & , & H_0 = \frac{1}{2}u^2. \end{array}$$

Take a $\mathcal{K} \in \mathcal{V}^1 \otimes \mathcal{V}^1$ with $\mathcal{K} = \sum_{|J| \leq 10} K^J D_J$, where $K^J = K^J(x, u_I)$, $|I| \leq 6$. The invariant subspace coincides with the subspace generated by the two Hamiltonian structures $\mathcal{D}$ and $\mathcal{E}$.

[6]This is due to the compatibility of the two structures. Cf. [Olv], Theorem 7.24. Since no use is made of the compatibility here, we do not dwell on this point here. See also the discussion following Theorem 7.27.

4.3.14 Example (Boussinesq equation)
The Boussinesq equation

$$\begin{aligned} u_t &= v_x \\ v_t &= \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x \end{aligned}$$

has a bi-Hamiltonian structure given by

$$\mathcal{D} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

$$H_1 = -\tfrac{1}{6}u_x^2 + \tfrac{4}{9}u^3 + \tfrac{1}{2}v^2,$$

$$\mathcal{E} = \begin{pmatrix} D_x^3 + 2uD_x + u_x & 3vD_x + 2v_x \\ 3vD_x + v_x & \tfrac{1}{3}D_x^5 + \tfrac{5}{3}(uD_x^3 + D_x^3 \cdot u) - (u_{xx}D_x + D_x \cdot u_{xx}) + \tfrac{16}{3}uD_x \cdot u \end{pmatrix},$$

$$H_0 = \tfrac{1}{2}v.$$

Take a $\mathcal{K} \in \mathcal{V}^1 \otimes \mathcal{V}^1$ with $\mathcal{K} = (\sum_{|J|\le 10} K^{\alpha\beta,J} D_J)$, where $K^{\alpha\beta,J} = K^{\alpha\beta,J}(x, u_I, v_I)$, $|I| \le 6$ and $\alpha, \beta = 1, 2$. Again the invariant subspace coincides with the subspace generated by the two Hamiltonian structures $\mathcal{D}$ and $\mathcal{E}$. (See Appendix A)

The striking fact, that the above invariant subspaces are so small, and that they are contained in the skew-symmetric subspace is a new nonclassic phenomenon and worth studying. To illustrate the strangeness of the above results, let us take an ordinary vector field X on a manifold M with $\dim M > 1$. Away from singularities X can be straightened, i.e. there exists a coordinate system $(x^1, \ldots, x^p)$, such that $X = \frac{\partial}{\partial x^1}$. Any locally defined differential geometric object (e.g. a tensor field), whose coefficients depend only on $(x^2, \ldots, x^p)$ is invariant under the flow of X. In particular, locally, the subspace of tensor fields of a given type, invariant under the flow of X is always infinite dimensional.

Appendix A

The Boussinesq Equation

The following pages include a MAPLE worksheet demonstrating how the package `jets` is used to find, up to a given order, all Hamiltonian structures of the Boussinesq equation (cf. Example 4.3.14).

```
> restart;
> with(Desolv):
> with(jets):
> ivar:=[t,x]; Ivar:=[x]; dvar:=[u,v]; var:=op(alljets(6,Ivar,dvar));
```

$$ivar := [t, x]$$

$$Ivar := [x]$$

$$dvar := [u, v]$$

$$var := x, u, v, u_{x}, v_{x}, u_{x, x}, v_{x, x}, u_{x, x, x}, v_{x, x, x}, u_{x, x, x, x}, v_{x, x, x, x}, u_{x, x, x, x, x}, v_{x, x, x, x, x}, u_{x, x, x, x, x, x}, v_{x, x, x, x, x, x}$$

```
> el:=a->Euler(a,Ivar,dvar);
  hm:=a->homotopy(a,Ivar,dvar)[1];
  hh:=a->Helmholtz(a,Ivar,dvar);
  vh:=a->vhomotopy(a,Ivar,dvar);
  bp:=a->intnorm(a,Ivar,dvar);
  hc:=b->a->hamchar(a,b,Ivar,dvar);
  dop:=a->diffop(a,Ivar,dvar);
```

$$el := a \rightarrow \mathrm{Euler}(a, Ivar, dvar)$$

$$hm := a \rightarrow \mathrm{homotopy}(a, Ivar, dvar)_{1}$$

$$hh := a \rightarrow \mathrm{Helmholtz}(a, Ivar, dvar)$$

$$vh := a \rightarrow \mathrm{vhomotopy}(a, Ivar, dvar)$$

$$bp := a \rightarrow \mathrm{intnorm}(a, Ivar, dvar)$$

$$hc := b \rightarrow a \rightarrow \mathrm{hamchar}(a, b, Ivar, dvar)$$

$$dop := a \rightarrow \mathrm{diffop}(a, Ivar, dvar)$$

The Boussinesq system of evolution equations:

```
> Eq:=[u[t]=v[x],v[t]=1/3*u[x,x,x]+8/3*u*u[x]];
```

$$Eq := \left[u_t = v_x, v_t = \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x\right]$$

The left hand side is a characteristic in the spatial variables only:

```
> P:=[v[x],1/3*u[x,x,x]+8/3*u*u[x]];
```

$$P := \left[v_x, \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x\right]$$

Check if P is a divergence:

```
> map(el,P);
```

$$[[0, 0], [0, 0]]$$

P is the divergence of:

```
> map(hm,P): EL:=[%[2],%[1]];
```

$$EL := \left[\frac{4}{3} u^2 + \frac{1}{3} u_{x,x}, v\right]$$

Is EL an Euler-Lagrange equation:

```
> hh(EL);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Compute the Lagrangian, and reduce its order through an integration by parts.
This is the Hamiltonian functional of the first Hamiltonian structure:

```
> vh(EL): H1:=bp(%);
```

$$H1 := \frac{4}{9} u^3 - \frac{1}{6} u_x^{\,2} + \frac{1}{2} v^2$$

Define a general functional bi-vector BB of order 10 and jet order 6:

```
> BB:=mkmat(map(a->map(b->map(c->[cat(Q,a,b,op(c))(var),c],map(symch,
  [$0..10],Ivar)),dvar),dvar)):
```

Compute the infinitesimal invariance condition $L_{v_P}(BB) = 0$:

```
> lsmp:=lddop20(P,BB,ivar,dvar):
> elem:=s->map(a->a[1],map(b->op(b),map(c->op(c),mklist(s)))):
```

Extract the differential conditions by equating the components to zero:

```
> cnd:=getcond(elem(lsmp),elem(BB),Ivar,dvar):
```

Solve the PDE system:

```
> jsolve(cnd): res:=subs(cnd[4],%):
> subs01(res[3],copy(BB),res[4]):
  smp:=map(a->map(gcollect,a,ivar),%);
```

$$smp := \left[\begin{bmatrix} 0 & [[1,[x]]] \\ [[1,[x]]] & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} [[3,[x,x,x]],[6\,u,[x]],[3\,u_x,[\;]]] & [[9\,v,[x]],[6\,v_x,[\;]]] \\ [[9\,v,[x]],[3\,v_x,[\;]]] & [[1,[x,x,x,x,x]],[10\,u,[x,x,x]],[15\,u_x,[x,x]],[9\,u_{x,x}+16\,u^2,[x]],[2\,u_{x,x,x}+16\,u\,u_x,[\;]]] \end{bmatrix} \right]$$

Normalize the output by hand:

```
> smp := [mkmat([[0, [[1, [x]]]], [[[1, [x]]], 0]]),
  map(a->mulcon(2,a,ivar),mkmat([[[[1/2, [x, x, x]], [u, [x]],
  [1/2*u[x], []]], [[3/2*v, [x]], [v[x], []]]], [[[3/2*v, [x]],
```

```
[1/2*v[x], []]], [[1/6, [x, x, x, x, x]], [5/3*u, [x, x, x]],
[5/2*u[x], [x, x]], [3/2*u[x,x]+8/3*u^2, [x]],
[1/3*u[x,x,x]+8/3*u*u[x], []]]]]))];
```

$$smp := \left[\begin{bmatrix} 0 & [[1,[x]]] \\ [[1,[x]]] & 0 \end{bmatrix}, \begin{bmatrix} [[1,[x,x,x]],[2\,u,[x]],[u_x,[\ ]]] & [[3\,v,[x]],[2\,v_x,[\ ]]] \\ [[3\,v,[x]],[v_x,[\ ]]] & \left[\left[\frac{1}{3},[x,x,x,x,x]\right],\left[\frac{10}{3}\,u,[x,x,x]\right],[5\,u_x,[x,x]],\left[3\,u_{x,x}+\frac{16}{3}\,u^2,[x]\right],\left[\frac{2}{3}\,u_{x,x,x}+\frac{16}{3}\,u\,u_x,[\ ]\right]\right] \end{bmatrix} \right]$$

The skew adjoint operator for the first Hamiltonian structure:

```
> DD:=smp[1];
```

$$DD := \begin{bmatrix} 0 & [[1,[x]]] \\ [[1,[x]]] & 0 \end{bmatrix}$$

The Jacobi identity:

```
> chkjac(DD,ivar,dvar);
```

$$0$$

Or equivalently:

```
> nsbra3(DD,DD,ivar,dvar);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The infinitesimal invariance condition:

```
> lddop20(P,DD,ivar,dvar);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here is the first Hamiltonian structure of the Boussinesq equation:

```
> hc(DD)(H1); %-P;
```

$$\left[v_x, \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x\right]$$

$$[0, 0]$$

```
> hf1:=a->hamflow(a,H1,DD,ivar,dvar);
```

$$hf1 := a \to \mathbf{hamflow}(a, H1, DD, ivar, dvar)$$

This is the second Hamiltonian functional:

```
> H0:=1/2*v;
```

$$H0 := \frac{1}{2} v$$

The skew adjoint operator for the second Hamiltonian structure:

```
> EE:=smp[2];
```

$$EE :=$$

$$\begin{bmatrix} [[1, [x, x, x]], [2\,u, [x]], [u_x, [\ ]]] & [[3\,v, [x]], [2\,v_x, [\ ]]] \\ [[3\,v, [x]], [v_x, [\ ]]] & \left[\left[\frac{1}{3}, [x, x, x, x, x]\right], \left[\frac{10}{3} u, [x, x, x]\right], [5\,u_x, [x, x]], \left[3\,u_{x,x} + \frac{16}{3} u^2, [x]\right], \left[\frac{2}{3} u_{x,x,x} + \frac{16}{3} u\, u_x, [\ ]\right]\right] \end{bmatrix}$$

Check the Jacobi identity:

```
> chkjac(EE,ivar,dvar);
```

$$0$$

Or equivalently:

```
> nsbra3(EE,EE,ivar,dvar);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The infinitesimal invariance condition:

```
> lddop20(P,EE,ivar,dvar);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

They are compatible:

```
> nsbra3(DD,EE,ivar,dvar);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here is the second Hamiltonian structure of the Boussinesq equation:

```
> hc(EE)(H0); %-P;
```

$$\left[v_x, \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x\right]$$

$$[0, 0]$$

```
> hf0:=a->hamflow(a,H0,EE,ivar,dvar);
```

$$hf0 := a \to \mathrm{hamflow}(a, H0, EE, ivar, dvar)$$

The inverse of DD:

```
> DD_:=_a->[hm(_a[2]),hm(_a[1])];
```

$$DD\_ := \_a \to [\mathrm{hm}(\_a_2), \mathrm{hm}(\_a_1)]$$

The Lenard recursion operator coming from the bi-Hamiltonian structure of the Boussinesq equation:

```
> RR:=a->dop(EE)(DD_(a)):
```

Get the symmetry characteristics using the recursion operator:

```
> Q0:=[0,0];
```

$$Q0 := [0, 0]$$

```
> Q1:=P;
```

$$Q1 := \left[v_x, \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x \right]$$

```
> Q2:=RR(Q1);
```

$$Q2 := \left[\frac{25}{3} u_x u_{x,x} + \frac{10}{3} u\, u_{x,x,x} + \frac{1}{3} u_{x,x,x,x,x} + \frac{20}{3} u^2 u_x + 5\, v\, v_x, \right.$$
$$\left. \frac{5}{3} v\, u_{x,x,x} + \frac{40}{3} v\, u\, u_x + \frac{20}{3} v_x u^2 + \frac{10}{3} v_x u_{x,x} + \frac{1}{3} v_{x,x,x,x,x} + \frac{10}{3} u\, v_{x,x,x} + 5\, u_x v_{x,x} \right]$$

```
> _Q0:=Q0;
```

$$\_Q0 := [0, 0]$$

```
> _Q1:=[u[x],v[x]];
```

$$\_Q1 := [u_x, v_x]$$

```
> _Q2:=RR(_Q1);
```

$$\_Q2 := \left[v_{x,x,x} + 4\, u\, v_x + 4\, u_x v, 4\, v\, v_x + \frac{1}{3} u_{x,x,x,x,x} + 4\, u\, u_{x,x,x} + 8\, u_x u_{x,x} + \frac{32}{3} u^2 u_x \right]$$

Characteristics and conserved densities for the first Hamiltonian structure (Olver pp. 460/461):

```
> H0:=1/2*v; Q0;
```

$$H0 := \frac{1}{2} v$$

$$[0, 0]$$

```
> H1:=(bp@vh@DD_)(Q1); Q1;
```

$$H1 := \frac{4}{9} u^3 - \frac{1}{6} u_x^{\,2} + \frac{1}{2} v^2$$

$$\left[v_x, \frac{1}{3} u_{x,x,x} + \frac{8}{3} u\, u_x\right]$$

```
> H2:=(bp@vh@DD_)(Q2); Q2;
```

$$H2 := \frac{20}{9} v\, u^3 - \frac{10}{3} u\, u_x\, v_x - \frac{5}{6} v\, u_x^{\,2} + \frac{5}{6} v^3 + \frac{1}{3} v_{x,x}\, u_{x,x}$$

$$\left[\frac{25}{3} u_x\, u_{x,x} + \frac{10}{3} u\, u_{x,x,x} + \frac{1}{3} u_{x,x,x,x,x} + \frac{20}{3} u^2\, u_x + 5\, v\, v_x,\right.$$
$$\left.\frac{5}{3} v\, u_{x,x,x} + \frac{40}{3} v\, u\, u_x + \frac{20}{3} v_x\, u^2 + \frac{10}{3} v_x\, u_{x,x} + \frac{1}{3} v_{x,x,x,x,x} + \frac{10}{3} u\, v_{x,x,x} + 5\, u_x\, v_{x,x}\right]$$

Check if Qi is the characteristic of Hi:

```
> hc(DD)(H0)-Q0; hc(DD)(H1)-Q1; hc(DD)(H2)-Q2;
```

$$[0, 0]$$
$$[0, 0]$$
$$[0, 0]$$

Check if Hi is a conserved density:

```
> el(hf1(_H0)); el(hf1(_H1)); el(hf1(_H2));
```

$$[0, 0]$$
$$[0, 0]$$
$$[0, 0]$$

Other characteristics and conserved densities for the first Hamiltonian structure (Olver pp. 460/461):

```
> _H0:=u; _Q0;
```

$$\_H0 := u$$
$$[0, 0]$$

```
> _H1:=(bp@vh@DD_)(_Q1); _Q1;
```

$$\_H1 := u\,v$$

$$[u_x, v_x]$$

```
> _H2:=(bp@vh@DD_)(_Q2); _Q2;
```

$$\_H2 := \frac{8}{9}u^4 + 2\,u\,v^2 - 2\,u\,u_x^{\,2} + \frac{1}{6}u_{x,x}^{\,2} - \frac{1}{2}v_x^{\,2}$$

$$\left[v_{x,x,x} + 4\,u\,v_x + 4\,u_x\,v,\ 4\,v\,v_x + \frac{1}{3}u_{x,x,x,x,x} + 4\,u\,u_{x,x,x} + 8\,u_x\,u_{x,x} + \frac{32}{3}u^2\,u_x\right]$$

Check if _Qi is the characteristic of _Hi:

```
> hc(DD)(_H0)-_Q0; hc(DD)(_H1)-_Q1; hc(DD)(_H2)-_Q2;
```

$$[0, 0]$$

$$[0, 0]$$

$$[0, 0]$$

Check if _Hi is a conserved density:

```
> el(hf1(_H0)); el(hf1(_H1)); el(hf1(_H2));
```

$$[0, 0]$$

$$[0, 0]$$

$$[0, 0]$$

```
>
```

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