Linear differential elimination for analytic functions

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Abstract

This paper provides methods to decide whether a given analytic function of several complex variables is a linear combination of finitely many given analytic functions with coefficients of the following special form: Each one of these coefficients is a composition of an unknown analytic function of less arguments than the function to be expressed, with fixed analytic functions. Methods which compute suitable coefficient functions in the affirmative case are presented as well.

1 Introduction

The aim of this paper is to provide a method how to decide whether a given analytic function of n complex variables is linear combination of finitely many given analytic functions. As coefficients not only constants are allowed but each coefficient may be an arbitrary analytic function of specified analytic complex valued functions, the number of which must be less than n but may vary with the coefficient. In case it is such a linear combination, methods to find suitable coefficient functions are given.
As a small example one might ask whether the function $(\sin x)^2$ is a linear combination of the form $f_1(x) \cdot (\cos(x+y))^2 + f_2(y) \cdot \cos(2x+y)$ with analytic functions f_1, f_2 of only one variable, where x and y are the specified functions of two independent variables. Indeed, one may choose $f_1(x) = 1$ and $f_2(y) = -\cos(y)$ in order to represent $(\sin x)^2$ in that way, but how would one answer the question without knowing these coefficients?
Note, the case of unrestricted coefficient functions excluded here is the rather trivial case of one linear equation over a field of meromorphic functions. The case of constants as coefficient functions can be treated by conventional methods of linear algebra, if one looks at the equation together with its various derivatives to turn the given equation into a system of linear equations. For the more subtle intermediate situation treated here, this method does not work immediately, since forming derivatives also produces new unknown functions, namely the derivatives of the original unknown functions.
To explain the method we recall from an introductory course in analysis a typical proof of the identity $\exp(x+y) = \exp(x)\exp(y)$. We reformulate this by asking: Is $\exp(x+y)$ of the form $f(y)\exp(x)$? Note first: The latter functions are characterized by the linear differential equation $u_x(x,y) - u(x,y) = 0$. Secondly, note that $\exp(x+y)$ satisfies this equation and hence is of the form $f(y)\exp(x)$. Thirdly, to determine $f(y)$, note that

$f(y) = \exp(x+y)/\exp(x)$ for any x, hence also for $x = 0$ and the claim follows. These three steps will be appropriately generalized.
The problem of finding an implicit description for a vector space of functions as above, i. e. of finding characterizing linear pde's, is addressed in Section 2. In Section 3 we explain how an explicit representation of a given function in that vector space can be computed and alternatives to the methods of Section 2 are obtained as a byproduct. Finally, Section 4 discusses characterizing pde's of special forms and gives applications. For instance, one sees how to decide whether a function given in some coordinates of n-space can be written as a sum of functions depending on $n-1$ coordinates in a different (given) coordinate system. Also the question whether a map of a Lie group into $\mathbb{C}^n$ can be augmented to a matrix representation is discussed in a simple case, cf. Example 4.8.
Methodically apart from elementary linear algebra the formal theory of linear differential equations following Janet is applied, cf. [Jan 29], [PlR 05]. Our applications of this theory go in the opposite direction of [ClQ 08] in the sense that we start with functions and arrive at linear pde's, which we use for recognition.
Unlike usual texts on differential algebra, cf. e. g. [Kol 73], we shall deal with composite functions. In fact, we could not find any reference in differential algebra, where composition of functions is systematically treated.
In the context of computing conservation laws for pde's, problems similar to ours came up in [WBM 99]. When it is possible to give the general solution to a pde system in closed form, it will in general depend in a redundant way on arbitrary constants and arbitrary functions of certain of the independent variables. Ways of identifying this redundancy are discussed in [WBM 99, Section 4.1]. Linear expressions as dealt with below, in which, however, unknown functions depend on certain of the independent variables and not on given functions, are treated there using different methods. A collection of powerful routines for solving such problems is the software package CRACK, developed under REDUCE since the 1990s, cf. e. g. [Wol 04] for a short description.
The examples presented in the sequel were dealt with using an implementation of Janet's algorithm for linear pde's (with non-constant coefficients) in Maple [BCG 03], relying on efficient and more general techniques for computing involutive bases [BGY 01].

2 Implicitization of certain spaces of functions

Our general notation for this paper is as follows:

Definition 2.1. 1.) Let $\Omega \subset \mathbb{C}^n$ be open and connected. We denote by $K := \mathcal{M}(\Omega)$ the field of meromorphic functions on Ω and by R the ring $K\langle\partial_1, \ldots, \partial_n\rangle$ of differential operators, which is the skew polynomial ring over K in the partial derivatives ∂_i with respect to coordinates z_i of $\mathbb{C}^n$.
2.) $g_1, \ldots, g_k$ are non-zero analytic $\mathbb{C}$-valued functions on Ω (not necessarily distinct). The tuple of all g_i is referred to as g.
3.) For each i, $1 \leq i \leq k$, there is a $\nu(i) < n$ with $\nu(i)$ (functionally independent) analytic functions $\alpha_{i,j} : \Omega \to \mathbb{C}$, $j = 1, \ldots, \nu(i)$, sometimes taken together as one analytic

function $\alpha_i : \Omega \to \mathbb{C}^{\nu(i)} : z \mapsto (\alpha_{i,1}(z), \ldots, \alpha_{i,\nu(i)}(z))$, such that the Jacobian has rank $\nu(i)$ throughout Ω. The k-tuple of the α_i is referred to as α.
4.) The analytic function $u : \Omega \to \mathbb{C}$ is called *(linearly) (α, g)-representable*, if there exist functions $f_i : \alpha_i(\Omega) \to \mathbb{C}$ such that $f_i \circ \alpha_i$ is analytic for $i = 1, \ldots, k$ and

$$(\dagger) \qquad u(z) = f_1(\alpha_1(z))g_1(z) + \cdots + f_k(\alpha_k(z))g_k(z)$$

for all $z \in \Omega$.

The f_i are necessarily analytic:

Lemma 2.2. *Let $F_i : \Omega \to \mathbb{C}$ be analytic for $i = 1, \ldots, k$ and factor over α_i. Then each function $f_i : \alpha_i(\Omega) \to \mathbb{C}$ with $F_i(z) = f_i(\alpha_i(z))$ is uniquely determined and analytic.*

Proof. Uniqueness of the f_i is obvious. Note, $\alpha_i(\Omega)$ is open in $\mathbb{C}^{\nu(i)}$. To prove analyticity of f_i, let $p_0 \in \Omega$. For each i, choose $n - \nu(i)$ independent variables $\beta_{i,\nu(i)+1}, \ldots, \beta_{i,n}$ from $z_1, \ldots, z_n$ complementing α_i to an analytic diffeomorphism $\kappa_i := (\alpha_i, \beta_{i,\nu(i)+1}, \ldots, \beta_{i,n})$ of some open neighborhood U_0 of p_0 in Ω onto $\kappa_i(U_0)$. We denote some analytic right inverse of the projection $\pi_i : \kappa_i(U_0) \to \alpha_i(U_0)$ by ι_i. Then the restriction of f_i to $\alpha_i(U_0) \subseteq \alpha_i(\Omega)$ is given by $F_i \circ \kappa_i^{-1} \circ \iota_i$, and hence is analytic. By uniqueness, f_i is analytic on Ω. $\square$

Unfortunately, the (α, g)-representability of f is a rather strong requirement. It is easier to test representability on dense open subsets of Ω first.

Definition 2.3. u is *(α, g)-representable around* $p_0 \in \Omega$, if there is an open neighborhood $\Omega' \subset \Omega$ of p_0 such that $u_{|\Omega'}$ is $(\alpha_{|\Omega'}, g_{|\Omega'})$-representable. It is *essentially (α, g)-representable on* Ω, if the set of points $p \in \Omega$ around which it is representable, is dense (and open) in Ω.

Note, for an essentially (α, g)-representable function u there are two kinds of obstructions to (α, g)-representability in general: the function u may not be representable globally on Ω, but only in an open neighborhood of each point of Ω; secondly, there may exist points in Ω around which u is not representable no matter how small the neighborhood is chosen. Therefore, one might think of essential (α, g)-representability as both a local and a generic property.

For the example $n = 2$, $k = 1$, $\alpha_1(x, y) = \alpha_{1,1}(x, y) = x^2$, $g_1 = 1$, the function x is essentially (α, g)-representable on $\mathbb{C}^2 - \{(0, 0)\}$, but not globally. One may say, around a generic point of $\mathbb{C}^2$, the function x is (α, g)-representable, but non-trivial monodromy prevents extension to all of $\mathbb{C}^2$. We shall concentrate on essential representability.

Remark 2.4. The technique used in the proof of Lemma 2.2 can be applied as well to characterize essential representability of an analytic function u in terms of linear pde's for u. Around any point of Ω, α_i can be complemented by $n - \nu(i)$ independent variables $\beta_{i,\nu(i)+1}, \ldots, \beta_{i,n}$ from $z_1, \ldots, z_n$ to an analytic diffeomorphism $\beta = (\alpha_i, \beta_{i,\nu(i)+1}, \ldots, \beta_{i,n})$. If u is essentially (α_i, g_i)-representable, then in the new coordinates u/g_i is independent of the last $n - \nu(i)$ coordinates leading to $n - \nu(i)$ first order linear pde's for u with coefficients in K after passing back to the original coordinates. (Recall that K is the field of meromorphic functions on Ω so that division by $g_i \neq 0$ is always well-defined.) The converse is also clear. Note, since we deal with essential representability, the subsets of smaller dimension where leading coefficients vanish and hence Riquier's existence theorem cannot be applied, may be neglected.

In what follows, for $i = 1, \ldots, k$, we denote by $I(\alpha_i, g_i)$ the left ideal of $R = K\langle \partial_1, \ldots, \partial_n \rangle$ generated by $\sum_r B_{rs}\partial_r \circ \frac{1}{g_i(z)}$ for $1 \leq s \leq n - \nu(i)$, where the columns of $B \in K^{n\times(n-\nu(i))}$ form a basis for the nullspace of the Jacobian matrix

$$\left(\frac{\partial \alpha_{i,j}}{\partial z_r}\right)_{1\leq j\leq \nu(i),\, 1\leq r\leq n}.$$

We arrive at the following necessary and sufficient criterion for essential representability of an analytic function u as $f_i(\alpha_i(z))g_i(z)$ with fixed i.

Lemma 2.5. *For each $i = 1, \ldots, k$, the analytic function $u : \Omega \to \mathbb{C}$ is essentially (α_i, g_i)-representable if and only if u is annihilated by $I(\alpha_i, g_i)$.*

Proof. Applying the directional derivatives $\sum_r B_{rs}\partial_r$ for $s = 1, \ldots, n - \nu(i)$, where B is defined as above, to u/g_i and using the chain rule, we get first order linear pde's for u, which characterize essential (α_i, g_i)-representability. The left ideal $I(\alpha_i, g_i)$ is generated by these $n - \nu(i)$ linear pde's. Note, these pde's are globally defined on an open dense subset of Ω. □

As a consequence we obtain the first main result. We denote by $I(\alpha, g)$ the intersection of the left ideals $I(\alpha_i, g_i)$, $i = 1, \ldots, k$.

Theorem 2.6. *The analytic function $u : \Omega \to \mathbb{C}$ is essentially (α, g)-representable if and only if u is annihilated by $I(\alpha, g)$.*

Proof. By the duality between systems of linear pde's and their solution spaces, sums and intersections of left ideals of linear pde's correspond respectively to intersections and sums of the solution spaces. □

For later use we introduce the notation $I_S(\alpha, g) := \bigcap_{i\in S} I(\alpha_i, g_i)$ for any $S \subseteq \{1, \ldots, k\}$.

Unfortunately, computing these intersections of left ideals is not as fast as one might wish from a computational point of view. We therefore develop an algorithm to find defining equations. The key to this method is a matrix version of the original equation (†) and its various derivatives. For instance, (†) can be rewritten as

$$u(z) = \Delta_0(\alpha, g) \cdot (f_1(\alpha_1(z)), \ldots, f_k(\alpha_k(z)))^{tr} \qquad (\dagger_0)$$

with

$$\Delta_0(\alpha, g) := (g_1(z), \ldots, g_k(z)) \in K^{1\times k}.$$

In order to get a formal procedure for the various partial derivatives of ($\dagger_0$), we denote the jet variables for u and its partial derivatives by u_μ with multi-indices $\mu \in (\mathbb{Z}_{\geq 0})^n$, i. e. u_0 for u, $u_{(1,0,\ldots,0)}$ for the partial derivative $\partial_1 u$ of u w.r.t. its first argument, and so on. We form their sequence $(u_0, u_{(1,0,\ldots,0)}, u_{(0,1,0,\ldots,0)}, \ldots, u_{(0,\ldots,0,d)})$ in some ordering up to order $|\mu| := \mu_1 + \cdots + \mu_n = d$, so that the sequence for order i is an extension of that for order $i-1$, $i = 1, \ldots, d$, and denote the corresponding column vector by $j_d(u)$.

On the side of the $f_i \circ \alpha_i$ denote the jet variables by $f_{i,\mu}$ with $\mu \in (\mathbb{Z}_{\geq 0})^{\nu(i)}$. So $f_{i,0}$ stands for $f_i \circ \alpha_i$ and $f_{i,(1,0,\ldots,0)}$ for $(\partial_1 f_i) \circ \alpha_i$, etc.. The definition of $j_d(\alpha, f)$ is now clear starting with $j_0(\alpha, f) = (f_{1,0}, \ldots, f_{k,0})^{tr}$, then complementing by the jets of order d to obtain $j_d(\alpha, f)$ from $j_{d-1}(\alpha, f)$. So one obtains from ($\dagger_0$) by taking all partial derivatives up to order d the equation

$$(\dagger_d) \qquad j_d(u) = \Delta_d(\alpha, g) \cdot j_d(\alpha, f).$$

The formal procedure to define $\Delta_d(\alpha, g) \in K^{\zeta(d) \times \sigma(d)}$ with

$$\zeta(d) = \binom{n+d}{d}, \qquad \sigma(d) = \sum_{i=1}^{k} \binom{\nu(i)+d}{d}$$

for general $d \in \mathbb{N}$ recursively, is an obvious consequence of the product and the chain rule. Note, the order of the jets subdivides the matrix into blocks $\Delta_d^{(i,j)}(\alpha, g)$ with $0 \leq i, j \leq d$, where i and j refer to the order of the derivatives of u and f respectively. Each (i,j)-block $\Delta_d^{(i,j)}(\alpha, g)$ with $j > i$ is zero, and the recursion shows that an arbitrary $\Delta_d^{(i,j)}(\alpha, g)$ is computable from $\Delta_d^{(i-1,j)}(\alpha, g)$ and $\Delta_d^{(i-1,j-1)}(\alpha, g)$ similarly to the Pascal triangle. Cf. Examples 2.11 or 3.4 for concrete examples.

Remark 2.7. For any R-submodule M of R denote by $M_{\leq d}$ the K-subspace of all elements of order $\leq d$ in the natural filtration. One has

$$\zeta(d) - \operatorname{rank}(\Delta_d(\alpha, g)) = \dim I(\alpha, g)_{\leq d}.$$

Proof. Clearly,

$$\{z \in K^{1 \times \zeta(d)} \mid z \cdot \Delta_d(\alpha, g) = 0\} \to I(\alpha, g)_{\leq d} : z \mapsto z \cdot j_d(u)$$

is an isomorphism of K-vector spaces. □

Of course, it would be desirable to have access to the column rank of $\Delta_d(\alpha, g)$ in terms of α, g, so that the previous remark could be used to decide whether one has generators for $I(\alpha, g)$. Note, any (α, g)-representation of the zero function gives rise to a linear dependency of the columns of $\Delta_d(\alpha, g)$.

Lemma 2.8. *Let J_K be a Janet basis (with respect to a term over position ordering respecting the differential order) for the submodule of the free left R-module $R^k := \bigoplus_{i=1}^k Ru_i$ generated by $\bigoplus_{i=1}^k I(\alpha_i, g_i)$ and $u_1 + \ldots + u_k$. Then one has:*

$$\operatorname{rank}(\Delta_d(\alpha, g)) \leq \sigma(d) - \tau(d),$$

where $\tau(d)$ is the sum of the first $d+1$ Taylor coefficients of the Hilbert series $H_K(t)$ of J_K, i. e. the Taylor coefficient of t^d in $H_K(t)/(1-t)$.

Proof. Clearly the set of solutions of J_K is equal to

$$S(\alpha, g) := \{(g_1 \cdot f_1 \circ \alpha_1, \ldots, g_k \cdot f_k \circ \alpha_k) \mid f_i \text{ analytic}, g_1 \cdot f_1 \circ \alpha_1 + \ldots + g_k \cdot f_k \circ \alpha_k = 0\}$$

in a suitable neighborhood of an arbitrary point in a dense subset of Ω, and one has a linear map

$$\begin{array}{rcl} j_{d,g} : S(\alpha, g) & \rightarrow & \{s \in K^{\sigma(d)\times 1} \mid \Delta_d(\alpha, g) \cdot s = 0\} : \\ (u_1, \ldots, u_k) & \mapsto & j_d((u_1/g_1, \ldots, u_k/g_k)), \end{array}$$

whose image obviously has (generically) dimension $\tau(d)$. Since $\Delta_d(\alpha, g)$ has $\sigma(d)$ columns, the claim follows. □

Note, J_K is relatively cheap to compute, in comparison to the computation of a Janet basis for $\bigcap_{i=1}^k I(\alpha_i, g_i)$, which involves elimination. Of course, one would really prefer to have equality in the last remark. But unfortunately there are examples, where one has a proper inequality. The next lemma can often be used as a stopping criterion when computing a Janet basis of a characterizing pde system for the (α, g)-representable functions.

Lemma 2.9. *Let J_d denote the Janet basis of the left ideal generated by $I(\alpha, g)_{\le d}$ (corresponding to the linear dependencies of the rows of $\Delta_d(\alpha, g)$) and J the Janet basis for $I(\alpha, g)$. If the Hilbert series of J_d is denoted by $H_d(t)$, then one has:*
1.) If the coefficient $h_{d,j}$ of t^j in $H_d(t)/(1-t)$ satisfies

$$h_{d,j} > \sigma(j) - \tau(j)$$

for one $j > d$, then $J_d \ne J$.
2.) If

$$h_{d,j} = \sigma(j) - \tau(j)$$

for all $j > d$, then $J_d = J$.

Proof. Note that $h_{d,j} \ge h_{d+1,j} \ge \cdots \ge h_{d+i,j} = h_j$ for sufficiently big i and all j, where h_j is the coefficient of t^j in $H(t)/(1-t)$ with $H(t)$ the Hilbert series of J. In other words, we have $h_j = \dim R_{\le j}/I(\alpha, g)_{\le j} = \zeta(j) - \dim I(\alpha, g)_{\le j} = \mathrm{rank}(\Delta_j(\alpha, g))$. Now both claims follow from the last remark. □

Unfortunately, 2.) in Lemma 2.9 is not always applicable, since the inequation in Lemma 2.8 might be strict. $(g, \alpha) = ((1,1), \alpha_1 := (x+y, x+z), \alpha_2 := (x^2+y, z))$ gives a counterexample, whereas for the simpler system with (g, α) given by $((1,1), \alpha_1 := (x+y, x+z), \alpha_2 := (x^2+y))$ it becomes applicable. Now we present an algorithm which computes a Janet basis for $\langle I(\alpha, g)_{\le d}\rangle$. Whether or not it is already a Janet basis for

$I(\alpha, g)$ one has to check: If Lemma 2.9 1.) is violated, one has moral certainty that one is done, otherwise one has to increase d. If 1.) is violated without 2.) being applicable, one has to take a closer look, for instance by the methods of the next section.

In the following algorithm it is important to have a book-keeping device for the rows of the relevant matrices, e. g. at the beginning for $\Delta_d(\alpha, g)$, where the rows are in correspondence with the jet variables for u. If linear combinations of rows are taken or rows are omitted, the same is done for the corresponding jet variables resp. for the names build up from the jet variables via the linear combinations. We simply refer to this sort of parallel computing in the following algorithm as *named computing*. Another piece of notation needed in the algorithm is this: At various stages we have Janet bases J for ideals of $R = K\langle\partial_1, \ldots, \partial_n\rangle$ contained in $I(\alpha, g)$. The matrix obtained from $\Delta_d(\alpha, g)$ by taking only those rows corresponding to parametric derivatives of u, i. e. those partial derivatives of u which are not leading derivatives of linear consequences of J, will be called $\Delta_{d,J}(\alpha, g)$ with blocks $\Delta_{d,J}^{(i,j)}(\alpha, g)$ according to the differential orders as before.

Algorithm 2.10. (pde's for essential (α, g)-representability)

Input: (α, g) and order $d \in \mathbb{N}$.

Output: Janet basis of $\langle I(\alpha, g)_{\leq d}\rangle$.

Algorithm: **Step 1**: Set $i := 1$, $J := \{0\}$ as initial Janet basis.

Step (i, J): Compute a basis for the linear dependencies between the rows of $\Delta_{i,J}^{(i,i)}(\alpha, g)$ and take via named computing the corresponding linear combinations of the rows of the submatrix $(\Delta_{i,J}^{(i,0)}(\alpha, g), \ldots, \Delta_{i,J}^{(i,i-1)}(\alpha, g))$ of $\Delta_{i,J}(\alpha, g)$, i. e. the truncated last block row of $\Delta_{i,J}(\alpha, g)$. Apply the Gauß algorithm to the matrix $\Delta_{i-1,J}(\alpha, g)$ enlarged by the named rows just computed. Collect all names of resulting zero rows and update J by the Janet basis for the ideal of R generated by J and these names. If $i = d$, then output J, else proceed to Step $(i + 1, J)$ with the updated J.

The following example demonstrates the criterion of Lemma 2.9 and the algorithm in a very simple case.

Example 2.11. Let $u(x, y) = f_1(x+y^2)+f_2(x+y)$, so we deal with analytic functions that are defined on an open and connected subset of $\mathbb{C}^n$, where $n = 2$, and which are locally (around a generic point) expressible as linear combinations of $k = 2$ analytic functions of $\nu(1) = \nu(2) = 1$ argument composed with $\alpha_{1,1} = x + y^2$ and $\alpha_{2,1} = x + y$ respectively. The coefficients in these linear combinations are assumed to be $g_1 = g_2 = 1$.
Hence, the number of rows of $\Delta_d(\alpha, g)$ is the sum of the first $d + 1$ Taylor coefficients of the generating function $1/(1-t)^n$, which equals the d-th Taylor coefficient of $1/(1-t)^3 = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \ldots$.
This we have to compare to estimates of the column ranks of the matrices $\Delta_d(\alpha, g)$. To this end one has to check uniqueness of representations via the differential system $\{2yu_{(1,0)} - u_{(0,1)}, xv_{(1,0)} - yv_{(0,1)}, u + v\}$, where the first two equations characterize the first and the second summand respectively. Its solution space is of dimension 1 and

contains only constants. The number of columns of $\Delta_d(\alpha, g)$ is the sum of the first $d+1$ Taylor coefficients of $2/(1-t)$. So the column ranks of the matrices $\Delta_d(\alpha, g)$ are bounded above by the Taylor coefficients of the generating function $(2/(1-t)-1)/(1-t) = 1+3t+5t^2+7t^3+9t^4+\ldots$.
So we expect the first linear dependency of the rows of $\Delta_d(\alpha, g)$ for $d = 2$:

$$\Delta_2(\alpha, g) = \left(\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2y & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 4y^2 & 1 \\ 0 & 0 & 0 & 0 & 2y & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

The rows are indexed by u, $u_{(0,1)}$, $u_{(1,0)}$, $u_{(0,2)}$, $u_{(1,1)}$, $u_{(2,0)}$ resulting into the differential relation

$$(1) \qquad (y - \tfrac{1}{2})u_{(0,2)} + (-2y^2 + \tfrac{1}{2})u_{(1,1)} + (2y^2 - y)u_{(2,0)} - u_{(0,1)} + u_{(1,0)} = 0.$$

This is the Janet basis J of $\langle I(\alpha, g)_{\leq 2}\rangle$, and the number of partial derivatives of u of fixed order that are not leading derivatives of linear consequences of (1) is given by the Hilbert series $1+2t+2t^2/(1-t)$. Dividing this by $1-t$ gives by definition the generating function for the number of rows of $\Delta_{d,J}(\alpha, g)$, i. e. $1+3t+5t^2+7t^3+9t^4+\ldots$. Since this agrees with the estimates for the column ranks above, we are done by the criterion in Lemma 2.9: J is the Janet basis for the ideal $I(\alpha, g)$.

By substituting e. g. y^2 for u in (1) we can now easily decide whether the function y^2 is essentially (α, g)-representable: the result is $-1 \neq 0$, which answers the question in the negative. However, y^2-y+1 can be shown by the same method to be (α, g)-representable.

3 Computing a representation

In this section we shall outline some methods to find actual representations for representable functions. At the same time it will produce yet another method to compute generators for $I(\alpha, g)$. We start again from the equation $(\dagger_0)$. Let $\Gamma_0 \in K^{k\times 1}$ be a right inverse of $\Delta_0(\alpha, g)$, i. e. $\Delta_0(\alpha, g) \cdot \Gamma_0 = 1$, and assume $X_1, \ldots, X_{k-1} \in K^{k\times 1}$ form a basis of the kernel of the linear map induced by $\Delta_0(\alpha, g)$. Then equation $(\dagger_0)$ is equivalent to the existence of meromorphic functions $a_1(z), \ldots, a_{k-1}(z)$ on Ω such that

$$(2) \qquad j_0(\alpha, f) = u(z)\Gamma_0 + a_1(z)X_1 + \cdots + a_{k-1}(z)X_{k-1},$$

and the above equation is recovered from this by left multiplication of equation (2) by $\Delta_0(\alpha, g)$. Now Lemma 2.5, more precisely the special case $g_1 = 1$ of its proof, can be applied to each component of the right hand side of (2) to obtain linear pde's for the $a_i(z)$, which are locally necessary and sufficient for the solvability of $(\dagger)$. Denote the resulting linear pde system for the $u(z)$, $a_1(z)$, $\ldots$, $a_{k-1}(z)$ by $N(*)$.

Remark 3.1. Differential elimination (using a block ordering on the dependent variables) of the $a_i(z)$ in $N(*)$ is a new way to compute a Janet basis for $I(\alpha, g)$.

Example 3.2. Let

$$\begin{aligned} u(z) &= z_4 f_1(z_1+z_2, z_2+z_3) + z_1 f_2(z_2+z_3, z_4+z_1) + \\ & \quad z_2 f_3(z_3+z_4, z_1+z_2) + z_3 f_4(z_4+z_1, z_2+z_3), \end{aligned}$$

i. e. $g_1 = z_4$ and $\alpha_1 = (z_1+z_2, z_2+z_3)$, etc.. We have three possibilities to proceed for finding a characterizing pde system:
1.) via $\Delta_d(\alpha, g)$, cf. Algorithm 2.10;
2.) via intersections (with improvements), cf. Theorem 2.6;
3.) via differential elimination, cf. Remark 3.1.
Clearly the answer is the same in all three cases: The leading derivatives of the Janet basis elements are

$$u_{(2,0,0,0)},\ u_{(1,1,0,2)},\ u_{(1,1,1,1)},\ u_{(1,2,0,1)},\ u_{(1,1,2,0)},\ u_{(1,2,1,0)},$$
$$u_{(1,0,2,2)},\ u_{(0,2,1,2)},\ u_{(0,2,2,1)},\ u_{(0,3,1,1)}.$$

Note, the Janet basis has no elements of order 3, though one of order 2, five of order 4 and four of order 5.
Method 1.), being the fastest of the three methods, if no special tricks to speed up the computation were applied, needed about 50 minutes in Maple on a 2300 MHz Opteron machine. However, the stopping criterion Lemma 2.9 2.) was not applicable to prove that $\langle I(\alpha, g)_{\leq 5}\rangle = I(\alpha, g)$. So one gets an answer in a straightforward way, but does not have a proof.
Method 2.) applied in a naive way via $I_{\{1,2\}}$, $I_{\{1,2,3\}}$, $I_{\{1,2,3,4\}}$ did not finish within 500 minutes (where $I_{\{1,2\}}$ stands for $I_{\{1,2\}}(\alpha, g)$, etc.). However, one gets an answer via $I_{\{1,2\}}$, $I_{\{2,3\}}$, $I_{\{3,4\}}$, $I_{\{1,2,3\}} = I_{\{1,2\}} \cap I_{\{2,3\}}$, $I_{\{2,3,4\}} = I_{\{2,3\}} \cap I_{\{3,4\}}$, $I(\alpha, g) = I_{\{1,2,3\}} \cap I_{\{2,3,4\}}$ in 40 minutes. Here, of course, we had a proven result in the end.
Method 3.) also turns out to be not feasible if one tries to eliminate a_1, a_2, a_3 in one go. However, eliminating a_1 first, and then a_2 was quite quick, but the final a_3 was rather difficult: We had to use the linear differential analogon of degree steering as developed for the algebraic case in [PlR 08]. With these additional tricks, the complete computation time was below 30 minutes, and again there was a proven result in the end.
Modified example. If one replaces in the above version of $u(z)$ the first summand by the term $z_4 f_1(z_1+z_2, z_3+z_4)$ to make the situation more symmetric, one gets as leading derivatives of the Janet basis elements

$$u_{(2,0,0,0)},\ u_{(1,1,1,0)},\ u_{(1,2,0,0)},\ u_{(1,0,3,0)},\ u_{(0,2,2,0)}.$$

Not only the complexity of the Janet basis decreases considerably. The timings for method 1.) were 45 seconds, where again Lemma 2.9 2.) was not applicable to prove completeness, for method 2.) with the improved order of taking intersections 54 seconds and method 3.) without applying degree steering 67 seconds.

The main aim of the present section is to determine the coefficient functions f_i by starting from a specific $u(z)$, which we typically assume to be (α, g)-representable. Here (2) becomes a linear pde system for the $a_i(z)$ only. Whereas the procedure from the beginning of this section first solves for the $a_i(z)$, then via equation (2) for the $f_i \circ \alpha_i$, and then finally for the f_i, we outline a procedure now, which, for given $u(z)$, gives the f_i right away, or at least some easy to solve linear differential equations for each f_i individually in some order. It suffices to explain how to find f_1. To this end let $u(z)$ be explicitly given.

Remark 3.3. Let J_1 be a Janet basis for $I_{\{2,3,\ldots,n\}}(\alpha, g)$, whose elements are written as differential equations for $v = v(z)$ (rather than u to avoid confusion with the concrete u).
1.) Inserting both sides of (†) into the elements of J_1 yields a system of inhomogeneous linear differential equations for $f_1 \circ \alpha_1$ only. This system can also be obtained by inserting $g_1(z)f_1(\alpha_1(z)) - u(z)$ for $v(z)$ in J_1.
2.) The resulting system has the property that its equations are (by application of the chain rule) automatically written as linear combinations of compositions of partial derivatives of f_1 with α_1 and have functions in z as coefficients. The inhomogeneous part is obtained by inserting the concrete $u(z)$ for $v(z)$ into the equations of J_1.
3.) Write $\beta := \alpha_1$ and $m := \nu(1)$ for the number of its components. Since the components β_i of β are functionally independent one can around each point of a sufficiently big subset of Ω extend the β_i by certain of the original coordinates z_i to a new local coordinate system. Say the restrictions of $\beta_1, \ldots, \beta_m, z_{m+1}, \ldots, z_n$ form such a local coordinate system. Invert the function tuple, i. e. write $z_i = \gamma_i(\beta_1, \ldots, \beta_m, z_{m+1}, \ldots, z_n)$ for $i = 1, \ldots, n$. Insert these expressions for $z_1, \ldots, z_m$ into any of the coefficient functions and the right hand side of the equations of 2.).
4.) As a result of 3.) one has now linear inhomogeneous differential equations for f_1 valid for any parameters $z_{m+1}, \ldots, z_n$ ranging in a certain open subset of $\mathbb{C}^{n-m}$. In particular, one can take derivatives with respect to these parameters, one can specialize them, etc., to obtain more equations.
5.) The equations obtained in 4.) can be used to eliminate (preferably higher) derivatives of f_1 until one arrives at a contradiction (showing that $u(z)$ is not essentially representable) or to arrive at a rather easy to solve pde system for f_1.

In practice, point 3.) of the above remark might sometimes cause difficulties, since it involves solving nonlinear equations in principle. But usually one still has some helpful flexibility via the choices of the coordinates that have to be performed. It is also clear that at the end of the remark one has obtained a new system of the form (†) with a modified u, so that Remark 3.3 actually constitutes an algorithm for finding coefficient functions f_i. We demonstrate the method of Remark 3.3 by the following rather easy example.

Example 3.4. Consider $u(x, y) := xf_1(y+x^2)+yf_2(x+y^2)$ and try to express $u := x^3-y^3$. In this case $I(\alpha, g)$ is easily determined to have a Janet basis of two equations of order 3, which are satisfied by u. So u is expressible in this form, but we want to determine the coefficient functions f_1, f_2. We keep the terminology of Remark 3.3. The Janet basis J_1 consists of $v_{(1,0)}y^2 - \frac{1}{2}v_{(0,1)}y + \frac{1}{2}v$.

1. and 2.) Inserting $v = xf_1(y+x^2) - u$ into J_1 yields

$$(\tfrac{1}{2}yx - 2y^2x^2)f_1'(y+x^2) - (y^2 + \tfrac{1}{2}x)f_1(y+x^2) + \tfrac{1}{2}x^3 + y^3 + 3y^2x^2 = 0.$$

3.) Introduce new coordinates $\overline{x} := x, \overline{y} := y + x^2$, so that the left hand side of this equation is rewritten as

$$(\tfrac{1}{2}(\overline{y}-\overline{x}^2)\overline{x} - 2(\overline{y}-\overline{x}^2)^2\overline{x}^2)f_1'(\overline{y}) - ((\overline{y}-\overline{x}^2)^2 + \tfrac{1}{2}\overline{x})f_1(\overline{y}) + \tfrac{1}{2}\overline{x}^3 + (\overline{y}-\overline{x}^2)^3 + 3(\overline{y}-\overline{x}^2)^2\overline{x}^2.$$

4. and 5.) This equation holds for all $\overline{x}$, in particular for $\overline{x} = 0$, so that the equation $\overline{y}^3 - \overline{y}^2 f_1(\overline{y}) = 0$ remains. We obtain $f_1(\overline{y}) = \overline{y}$.
Note, to obtain f_2 in this case of only two summands, one simply has to insert $f_1(y+x^2)$ into the original equation $u = x(y+x^2) + yf_2(x+y^2)$ and solve for $f_2(x+y^2)$, which results in $f_2(x+y^2) = -(x+y^2)$.
In this particular example, there is an alternative way, which has not been discussed above. The matrix

$$\Delta_2(\alpha, g) = \left(\begin{array}{cc|cc|cc} x & y & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & x & 2\,y^2 & 0 & 0 \\ 1 & 0 & 2\,x^2 & y & 0 & 0 \\ \hline 0 & 0 & 0 & 6\,y & x & 4\,y^3 \\ 0 & 0 & 1 & 1 & 2\,x^2 & 2\,y^2 \\ 0 & 0 & 6\,x & 0 & 4\,x^3 & y \end{array}\right)$$

connecting $j_2(u)$ with $j_2(\alpha, f)$ is invertible, thus opening another possibility to express the f_i in terms of the concrete u and its partial derivatives. The general context, where this possibility of inverting $\Delta_d(\alpha, g)$ is applicable, is when one has equality in Lemma 2.9 2.) and the kernel is trivial, i. e. J_K is empty, $\tau(d) = 0$, and the coefficient functions are uniquely determined.

Remark 3.5. In the above discussion we assumed that the function $u(z)$ to be represented is concretely given. One can also lead the discussion in the case where it is not made explicit and obtains formulas for the f_i. These formulas are correct, when a representable $u(z)$ is chosen and incorrect, when a non-representable $u(z)$ is chosen. In principle, this can also be used as a criterion: If the formula leads to a wrong identity, $u(z)$ is not (α, g)-representable.

4 Examples and applications

The first question one might ask, is when are the (α, g)-representable functions characterized by the same pde system as the $(\tilde{\alpha}, \tilde{g})$-representable functions for some other pair $(\tilde{\alpha}, \tilde{g})$. Certainly a permutation of the summands has no effect. Also one summand might already be annihilated by the ideal for the sum of the others so that the omission of the corresponding (α_i, g_i) has no effect on the ideal. The following very simple example demonstrates the more serious effect which occurs when say $g_1 = \tilde{g}_1$ and α_1 and $\tilde{\alpha}_1$ at least locally factor over each other.

Example 4.1. Consider $u(x,y,z) = zf(x^2 + y, y^2 + x)$ on the one hand and $u(x,y,z) = zf(x,y)$ on the other hand. Both give rise to the same characterizing ideal generated by $zu_{(0,0,1)} - u$.

An obvious question is, when the characterizing linear pde system has constant coefficients. We start the discussion with a very easy example demonstrating already quite a few relevant points.

Example 4.2. Consider $u(x,y) = f_1(x) + f_2(x+y)\exp(x-y)$. Analytic functions of the form $f_1(x)$ are characterized as analytic solutions of $u_{(0,1)} = 0$. The second summand has the characterizing pde $u_{(1,0)} - u_{(0,1)} - 2u = 0$. Since both differential equations have constant coefficients, the corresponding differential operators D_1, D_2 commute, and the intersection of the principal ideals $I(\alpha_1, g_1)$ and $I(\alpha_2, g_2)$ equals their product, which is generated by the product $D_1 D_2$. The two lines in the plane $y = 0$ and $x - y - 2 = 0$ intersect in the point $(2,0)$ corresponding to the fact that one does not have unique (α, g)-representations: $0 = \exp(2x) - \exp(x+y)\exp(x-y)$.

Proposition 4.3. *Let the system (α, g) satisfy the following two conditions:*
1.) The g_i are constant or exponential functions whose arguments are polynomials of degree one.
2.) The $\alpha_{i,j}$ are polynomials of degree one.
Then $I(\alpha, g)$ can be generated by differential operators with constant coefficients.

Proof. Suppose all g_i are constant and 2.) is satisfied. Then the matrices $\Delta_d(\alpha, g)$ have only constant entries, and the linear dependencies of their rows generating $I(\alpha, g)$ can also be chosen to be written over $\mathbb{C}$. If for instance g_i is an exponential function whose argument is a polynomial of degree one, the columns of $\Delta_d(\alpha, g)$ corresponding to derivatives of f_i will be constant vectors multiplied by this exponential function. Clearly these exponential factors in the matrix $\Delta_d(\alpha, g)$ have no influence on the linear relations among the rows. □

In view of the discussion preceding Example 4.1 one cannot conclude from the constant coefficients in the generating set for $I(\alpha, g)$ that the $\alpha_{i,j}$ are necessarily polynomials of degree one. Which ideals in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ generate characterizing ideals $I(\alpha, g)$ for some system (α, g)? The answer is rather simple.

Proposition 4.4. *Let $I \trianglelefteq \mathbb{C}[\partial_1, \ldots, \partial_n]$. There exists a system (α, g) such that $\langle I \rangle_R = I(\alpha, g)$ if and only if I is a radical ideal such that each of its primary components is generated by polynomials of degree one.*

Proof. Assume $I \trianglelefteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ is a radical ideal such that each of its primary components is generated by polynomials of degree one. By Theorem 2.6 we may assume that I is already prime. If the generators are linear polynomials $p_s(\partial_1, \ldots, \partial_n)$, $s = 1, \ldots, m$, i. e. polynomials of degree one without constant term, choose a basis $(\beta_1, \ldots, \beta_{n-m})$ of the space of linear homogeneous polynomials in $\mathbb{C}[z_1, \ldots, z_n]$ annihilated by all $p_s(\partial_1, \ldots, \partial_n)$.

We can choose $g_1 := 1$ and in case $m < n$, $\alpha_1 := (\beta_1, \ldots, \beta_{n-m})$, whereas in case $m = n$ one has to choose $\nu(1) := 0$ and there is no α_1. If at least one of the generators $p_s(\partial_1, \ldots, \partial_n)$ has a constant term, one omits the constant terms to choose α_1 as just described. In this case g_1 can be chosen to be $\exp(a_1 z_1 + \cdots + a_n z_n)$ with $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying $p_s(a) = 0$ for $s = 1, \ldots, m$.
The converse follows from Proposition 4.3, Lemma 2.5, and Theorem 2.6. □

When can polynomial coefficients arise in generators of the characterizing ideal?

Proposition 4.5. *Let the system (α, g) satisfy the following two conditions:*
1.) The g_i are rational function multiples of exponential functions with rational function arguments.
2.) The $\alpha_{i,j}$ are rational functions.
Then $I(\alpha, g)$ can be generated by differential operators with polynomial coefficients.

Proof. The proof is similar to the one of Proposition 4.3: Under the hypothesis given, the entries of a given column of $\Delta_d(\alpha, g)$ are rational functions multiplied with the exponential function of a fixed rational function. For the linear dependencies of the rows the exponential factors are irrelevant: They can be chosen to have rational function coefficients or even polynomial function coefficients after multiplication with the common denominator. □

If the g_i are all equal to 1 and the $\alpha_{i,j}$ are polynomials of degree one, then the Janet basis of the left ideal $I(\alpha, g)$ consists of homogeneous elements. The following case is particularly simple: $u(x, y, z) = f_1(y, z) + f_2(x, z) + f_3(x, y)$ which has characterizing equation $u_{(1,1,1)} = 0$. So for instance, if one has to analyze a function of three variables, one can check whether it satisfies this equation to see whether the problem splits into three cases of functions of two variables. Or one has some pde system for a function of three variables and one is interested in solutions of the above form. Then one can add $u_{(1,1,1)} = 0$ to the system. Thomas' algorithm [Tho 37], which decomposes polynomial differential systems into so-called simple systems, lends itself as an adapted method for the treatment and simplification of these systems. Clearly there are variants of this splitting strategy. We would like to discuss one variant, where instead of x, y, z one has other coordinates of 3-space and wants to check whether a function given in x, y, z coordinates can be rewritten as a sum of three functions in two of the other coordinates. The next two examples are of this type, the first coming from spherical coordinates, the second from symmetric functions.

Example 4.6. The spherical coordinates r, θ, φ defined by $x = r\cos(\theta)\cos(\varphi)$, $y = r\cos(\theta)\sin(\varphi)$, $z = r\sin(\theta)$ give rise to three commuting differential operators

$$\begin{aligned} \partial_r &:= \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\partial_x + y\partial_y + z\partial_z), \\ \partial_\theta &:= \frac{1}{\sqrt{x^2+y^2}}\,(-xz\partial_x - yz\partial_y) + \sqrt{x^2+y^2}\,\partial_z, \\ \partial_\varphi &:= -y\partial_x + x\partial_y. \end{aligned}$$

The product of these three operators is a bit too lengthy to be reproduced here, but it allows a check whether a function in x, y, z can be written as a sum of functions each one depending only on two of r, θ, φ. Furthermore, the earlier discussion of characterizing systems with constant coefficients can be adjusted to some extent to constant coefficients in monomials in these three operators. Of course, the resulting coefficients in the $\partial_x, \partial_y, \partial_z$ are far from being constant.

The following example is taken from the context of invariants of finite groups. Since suitably chosen invariants can be viewed as coordinates of the orbit space, it can be understood as a variant of the above discussion.

Example 4.7. Let $e_1 := x + y + z$, $e_2 := yz + xz + xy$, $e_3 := xyz$ be the elementary symmetric functions in x, y, z. Consider $u(x, y, z) = f_1(e_2, e_3) + f_2(e_3, e_1) + f_3(e_1, e_2)$, i. e. in our earlier notation $g_1 = g_2 = g_3 = 1, \alpha_1 = (e_2, e_3)$ with cyclic permutation of the indices. The characterizing pde system for the functions u of the above form is given by a lengthy homogeneous differential equation of order 3. The associated differential operator D (suitably scaled) can be factored into three commuting operators of order 1:

$$\begin{aligned}\partial_1 &:= \frac{1}{\delta}(x^2(y-z)\partial_x + y^2(z-x)\partial_y + z^2(x-y)\partial_z),\\ \partial_2 &:= -\frac{1}{\delta}(x(y-z)\partial_x + y(z-x)\partial_y + z(x-y)\partial_z),\\ \partial_3 &:= \frac{1}{\delta}((y-z)\partial_x + (z-x)\partial_y + (x-y)\partial_z)\end{aligned}$$

with $\delta := (x-y)(x-z)(y-z)$ the Vandermonde polynomial, i. e. the skew-symmetric polynomial of lowest degree in x, y, z. One easily checks $\partial_i(e_j) = \delta_{i,j}$ so that ∂_i corresponds to the summand f_i. As a byproduct one has $D(e_1e_2e_3) = 1$. For instance, δ^2 is also symmetric of degree 6 and one might ask whether it is (α, g)-representable. In fact, it is not, because $D(\delta^2) = 18$. Hence $\delta^2 - 18e_1e_2e_3$ is (α, g)-representable.
One interesting property of D is the following:

$$D(e_1^{s_1}e_2^{s_2}e_3^{s_3}) = s_1s_2s_3e_1^{s_1-1}e_2^{s_2-1}e_3^{s_3-1},$$

so that D defines a filtration on the space of symmetric polynomials in x, y, z of which the (α, g)-representable ones form the bottom piece, those symmetric polynomials p for which $D(p)$ is (α, g)-representable form the second piece etc.. So by checking which power of D annihilates a symmetric polynomial p, one can determine the highest power of $e_1e_2e_3$ dividing the monomials in the e_i occurring in the expansion of p. In other words, the operator D has the same effect on the ring of symmetric polynomials as $\partial_x\partial_y\partial_z$ on $\mathbb{R}[x, y, z]$.

The next example is taken from representation theory of Lie groups. Given a map of a Lie group $\gamma : G \to \mathbb{C}^{n\times 1}$, is there a representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ such that $\gamma(g)$ is the first column of $\rho(g)$ for all $g \in G$? The example deals with the first row instead and takes $G := (\mathbb{R}, +)$ as a rather modest example of a Lie group.

Example 4.8. Is there a representation $\rho : \mathbb{R} \to \mathrm{GL}(3, \mathbb{C})$ of the Lie group $(\mathbb{R}, +)$ such that the first row of $\rho(x)$ is given by $\gamma(x) := (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ with explicit specification below?
Since $\rho(0)$ is the unit matrix, we should have $\gamma(0) = (1, 0, 0)$. Secondly, we should have

$$\rho_{1,1}(x + y) = (\rho(x)\, \rho(y))_{1,1},$$

i. e.

$$\gamma_1(x + y) = \gamma_1(x)\, \gamma_1(y) + \gamma_2(x)\, \rho_{2,1}(y) + \gamma_3(x)\, \rho_{3,1}(y).$$

In other words, we require the (α, g)-representability of the function $\gamma_1(x+y) - \gamma_1(x)\, \gamma_1(y)$ with $g := (\gamma_2(x), \gamma_3(x))$ and $\alpha = (\alpha_1, \alpha_2) := (y, y)$.
1.) Choose $\gamma(x) := (1 + x, x^2, x^3)$ as prescribed first row.
In this case the left ideal $I(\alpha, g)$ is generated by the left hand side of $x^2\, u_{(2,0)} - 4\, x\, u_{(1,0)} + 6\, u = 0$, but $\gamma_1(x + y) - \gamma_1(x)\, \gamma_1(y) = (1 + x + y) - (1 + x)\,(1 + y)$ does not satisfy this pde, and hence γ does not occur as a first row of a representation.
2.) Choose $\gamma(x) := (1 + 2\, x, -x, -x)$ as prescribed first row.
In this case $I(\alpha, g)$ is generated by the left hand side of $x\, u_{(1,0)} - u = 0$. This time the function $\gamma_1(x+y) - \gamma_1(x)\, \gamma_1(y) = (1+2\, x+2\, y) - (1+2\, x)\,(1+2\, y)$ satisfies the equation. However, the (α, g)-representation is not unique:

$$\rho_{21}(x) = x + x^2 + \xi(x), \quad \rho_{31}(x) = 3x - x^2 - \xi(x),$$

where $\xi(x)$ is arbitrary with $\xi(0) = 0$. The same method also yields the second and third column of $\rho(x)$, again with an arbitrary function coming in in each case. To discuss all possibilities certainly requires other methods as well, because the conditions for the remaining functions are nonlinear. But if one chooses $\xi(x)$ identically zero, the remaining entries are uniquely determined by standard commutative algebra techniques. The result is

$$\rho(x) = \begin{pmatrix} 1 + 2\, x & -x & -x \\ x + x^2 & 1 - \frac{1}{2}\, x^2 & -\frac{1}{2}\, x^2 \\ 3\, x - x^2 & -2\, x + \frac{1}{2}\, x^2 & 1 - 2\, x + \frac{1}{2}\, x^2 \end{pmatrix}.$$

References

[BCG 03] Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz. *The MAPLE Package "Janet": I. Polynomial Systems. II. Linear Partial Differential Equations.* In *Proc. of Computer Algebra in Scientific Computing CASC 2003*, V. G. Ganzha, E. W. Mayr, E. V. Vorozhtsov (eds.), 31–40 resp. 41–54. Garching, Germany: Institut für Informatik, TU München, 2003. Also available together with the package from `http://wwwb.math.rwth-aachen.de/Janet`.

[BGY 01] Y. A. Blinkov, V. P. Gerdt, D. A. Yanovich. *Construction of Janet bases, II. Polynomial Bases.* In *Computer Algebra in Scientific Computing CASC 2001*, V. G. Ganzha, E. W. Mayr, E. V. Vorozhtsov (eds.), Springer, 2001, 249–263.

[ClQ 08] T. Cluzeau, A. Quadrat. *Factoring and decomposing a class of linear functional systems.* Linear Algebra Appl. **428**:1 (2008), 324–381.

[Jan 29] M. Janet. *Leçons sur les systèmes d'équations aux dérivées partielles*, Cahiers Scientifiques IV, Gauthiers-Villars, Paris, 1929.

[Kol 73] E. R. Kolchin. *Differential algebra and algebraic groups.* Pure and Applied Mathematics, Vol. 54. Academic Press, New York-London, 1973.

[PlR 05] W. Plesken, D. Robertz. *Janet's approach to presentations and resolutions for polynomials and linear pdes.* Arch. Math. **84**:1 (2005), 22–37.

[PlR 08] W. Plesken, D. Robertz. *Elimination for coefficients of special characteristic polynomials.* Experiment. Math. **17**:4 (2008), 499–510.

[Tho 37] J. M. Thomas. *Differential Systems.* AMS Colloquium Publications, Vol. XXI, 1937.

[WBM 99] T. Wolf, A. Brand, M. Mohammadzadeh. *Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions.* J. Symbolic Comput. **27**:2 (1999), 221–238.

[Wol 04] T. Wolf. *Applications of CRACK in the classification of integrable systems.* In *Superintegrability in classical and quantum systems*, CRM Proc. Lecture Notes, Vol. 37, Amer. Math. Soc., Providence, RI, 2004, 283–300.