Thomas Decomposition of Algebraic and Differential Systems

Thomas Bächler[1], Vladimir Gerdt[2], Markus Lange-Hegermann[1], Daniel Robertz[1]

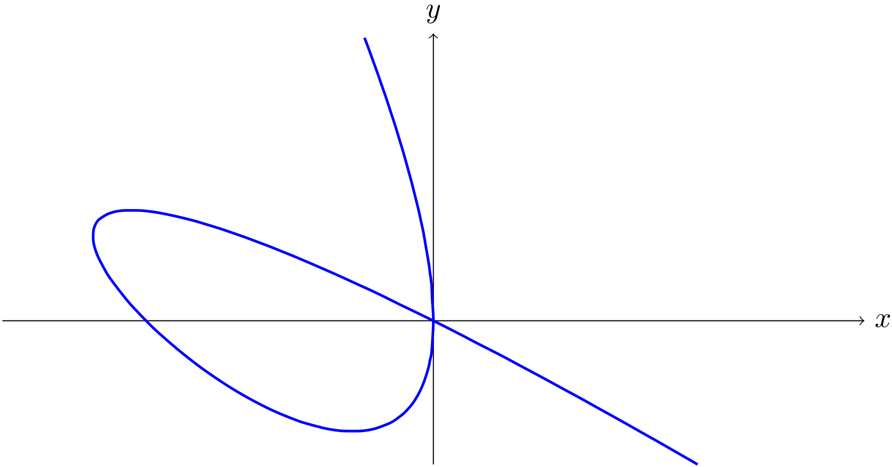
[1]Lehrstuhl B für Mathematik, RWTH Aachen University
[2]Joint Institute for Nuclear Research, Dubna

Outline

Motivating Example

Decompose the variety

$$x^3 + 3x^2y + 3xy^2 + y^3 + x^2 + 2xy = 0$$



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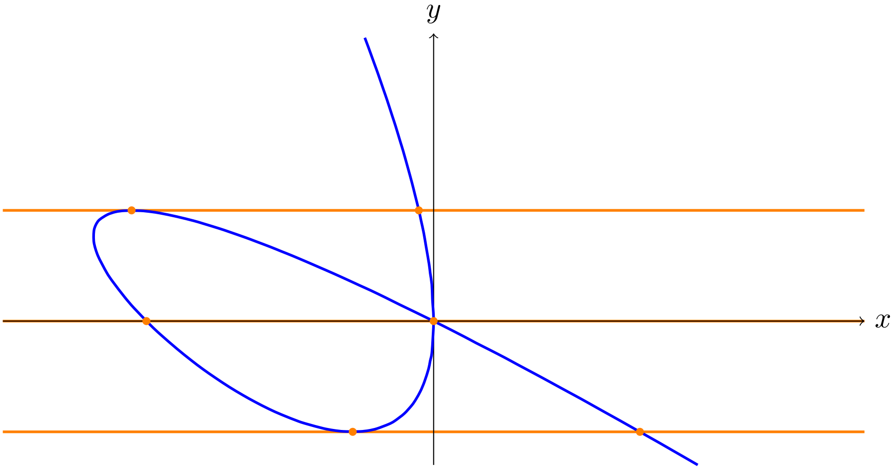

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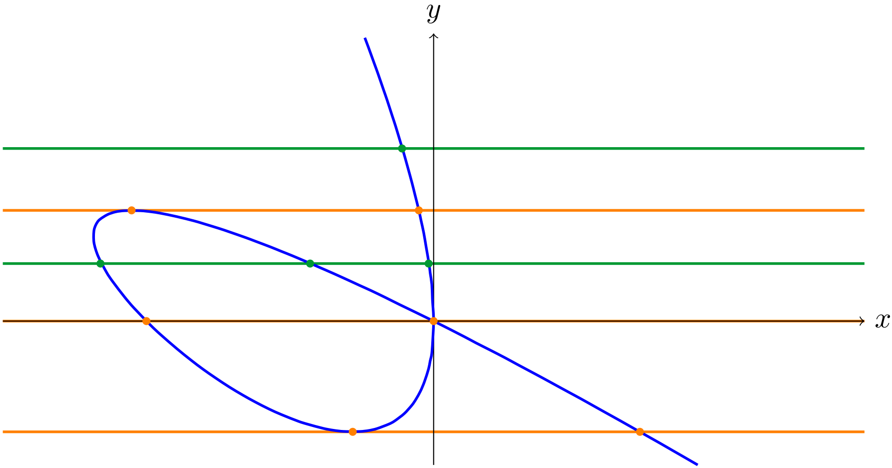

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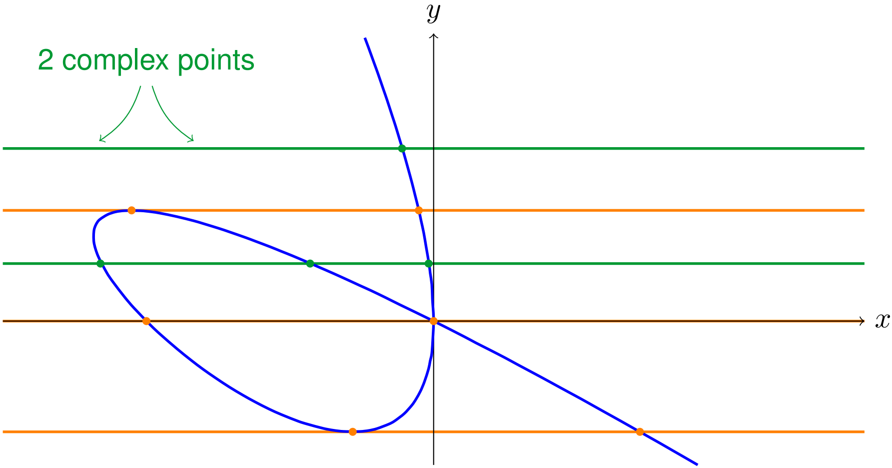
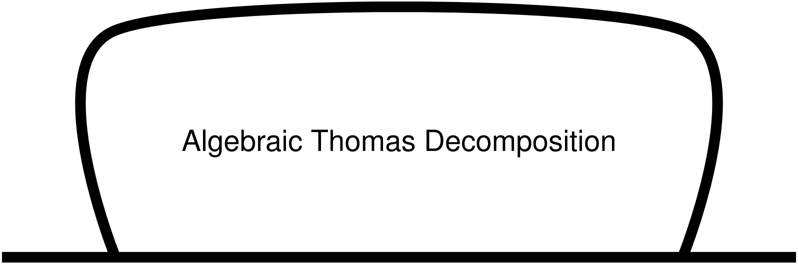


History

- **1937:** Introduced by Joseph Miller Thomas in his book "Differential Systems".
- **1998:** Implementation by Dongming Wang for the algebraic case.
- **2007:** Gerdt: "On decomposition of algebraic PDE systems into simple subsystems".
- **2009:** Plesken: "Counting solutions of polynomial systems via iterated fibrations".
- **2009:** Implementation by Bächler and Lange-Hegermann for the algebraic and ordinary/partial differential case.


Algebraic Thomas Decomposition

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  - **Triangular**
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    - The **leading coefficient** does not vanish.
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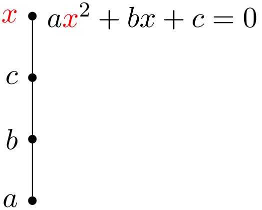
- **Algebraic THOMAS decomposition**: *Disjoint* decomposition into simple systems.

Simple Example

Example: $\{ax^2 + bx + c = 0\} \subseteq \mathbb{Q}[x, c, b, a]$.

Thomas decomposition w.r.t. the ordering $a < b < c < x$:



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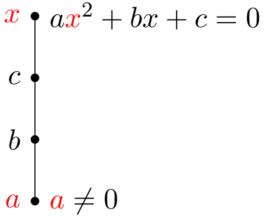

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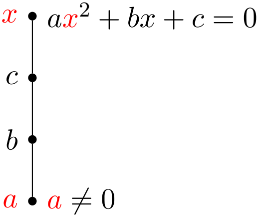

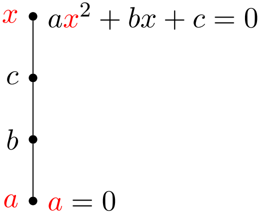

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Thomas decomposition w.r.t. the ordering $a < b < c < x$:

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c •
b •
a • $a \neq 0$

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	First system		Second system
x	$ax^2 + bx + c = 0$	x	$bx + c$
c	$4ac - b^2 \neq 0$	c	
b		b	
a	$a \neq 0$	a	$a = 0$

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 & & \quad \downarrow\ y \cdot q_1 - p_1 \\
y & p_2 = y^2 + 1 = 0 & y^2 + 1 = q_2
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$$q_2 - p_2$$

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Decide vanishing of a polynomial on all solutions of the system.

Subresultants

During the algorithm we need to

- Replace several equations by their gcd.
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Split the system using these conditions.

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Inequations are treated as an integral part of the system.

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Example

$$p := x^3 + 3x^2y + 3xy^2 + y^3 + x^2 + 2xy$$

Thomas Decomposition of $\{p = 0\}$, $x > y$:

$$S_1 := \{p = x^3 + \ldots = 0, \quad \mathrm{disc}_x(p) = y^3 + \ldots \neq 0\}$$
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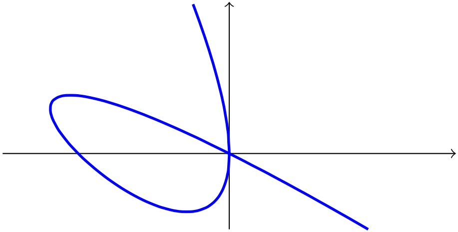

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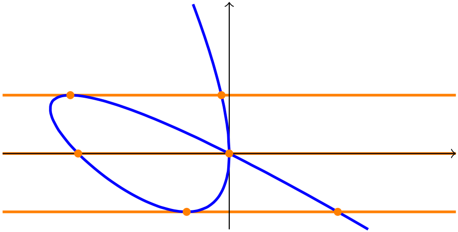

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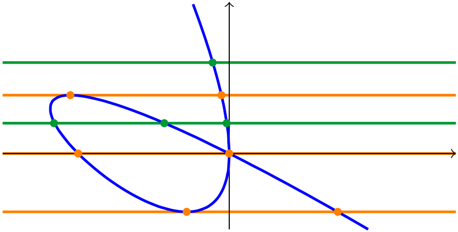

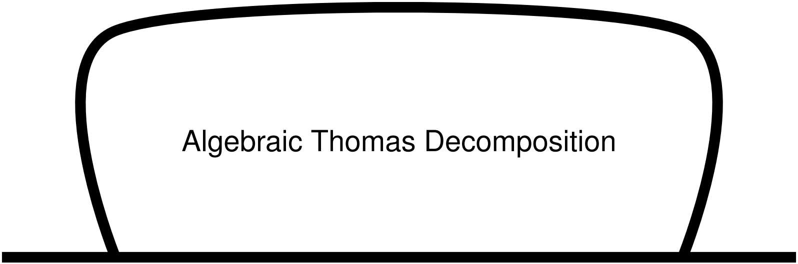
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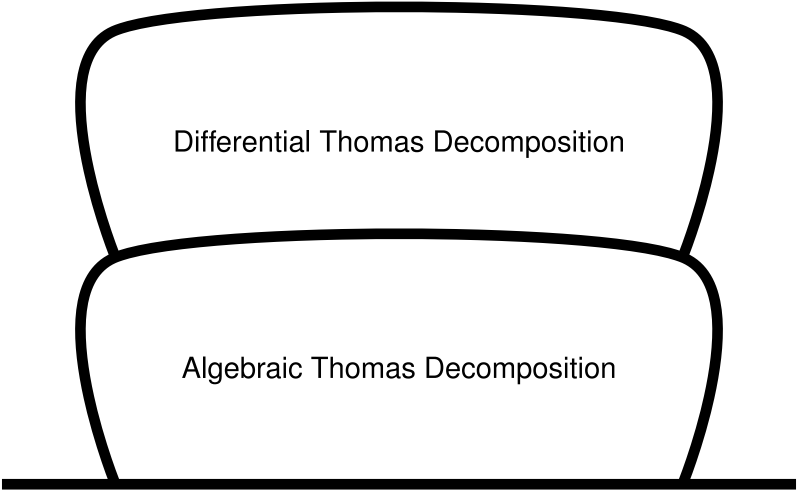
Differential Thomas Decomposition
Algebraic Thomas Decomposition

Differential Algebra

- **Setting:**
  - Dependent variables $U = \{u^{(1)}, \ldots, u^{(m)}\}$ ("functions")
  - Independent variables $\{x_1, \ldots, x_n\}$ and derivations $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$
  - A differential field F with $\mathrm{char}(F) = 0$

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- **Differential polynomial ring:**

$$F\{U\} = F[\underbrace{u_{\mathbf{i}}^{(j)}}_{\substack{\text{differential variables} \\ (\infty\text{-many})}} \mid j \in \{1 \dots m\}, \mathbf{i} \in \mathbb{Z}_{\geq 0}^n]$$

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- Differential equations $\rightsquigarrow F\{U\}$, e.g. Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \rightsquigarrow u_{0,1} + u_{0,0}u_{1,0} \in F\{U\}$$

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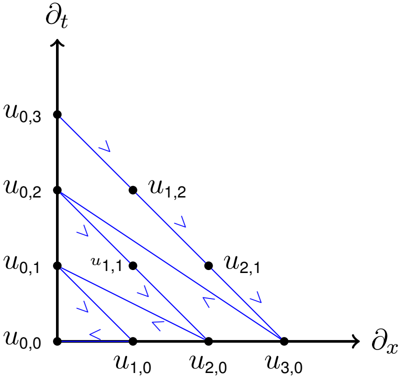
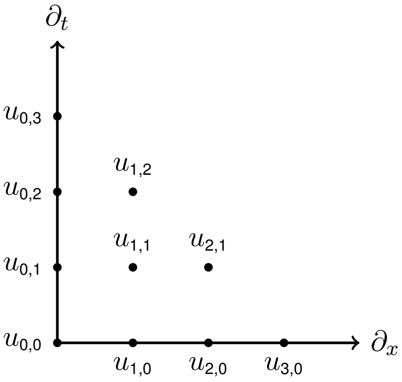
Use ranking '$<$' on the variables of $F\{U\}$ compatible with derivations.
Example for a ranking: $u_{0,0} < u_{1,0} < u_{0,1} < u_{2,0} < u_{1,1} < u_{0,2} < u_{3,0} < \ldots$



Differential Consequences

We indicate an equation in the picture by attaching it to its leader:

$u_{0,1} + u_{0,0}u_{1,0}$



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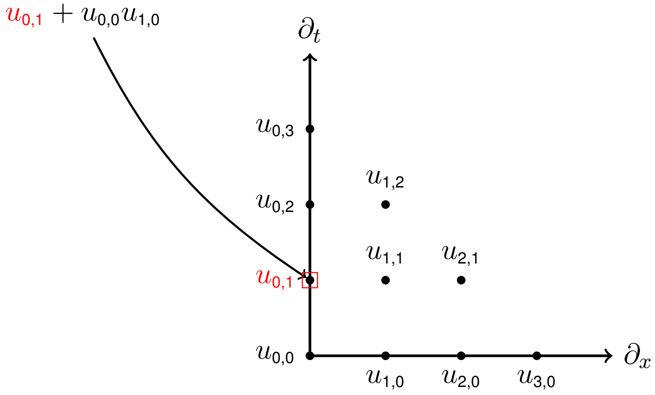
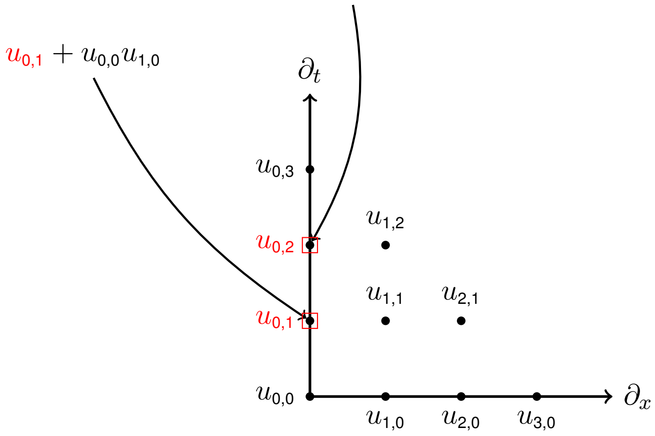


Differential Consequences

A differential equation also affects the derivatives of the leader:

$$\partial_t(u_{0,1} + u_{0,0}u_{1,0}) = u_{0,2} + u_{0,1}u_{1,0} + u_{0,0}u_{1,1}$$



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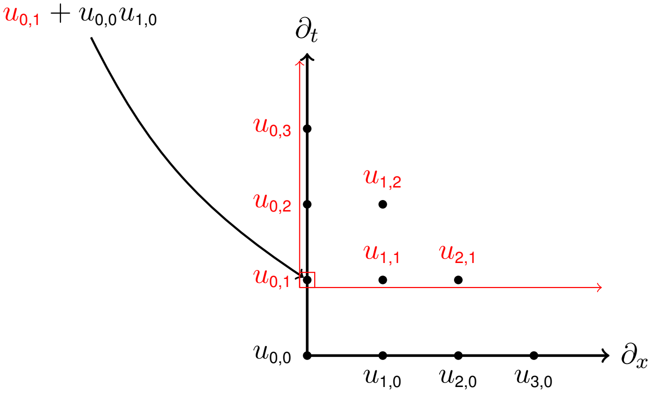
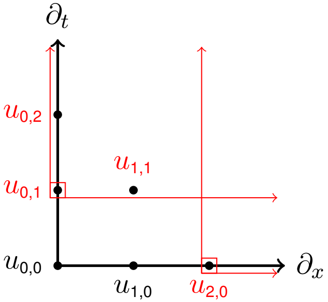


Differential Consequences

Add $u_{2,0} = 0$



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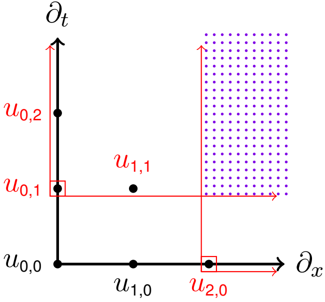
The second differential equation destroys triangularity:



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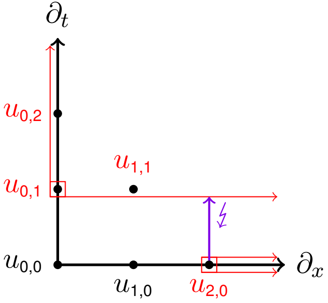
Use the involutive approach (Janet), which forbids certain derivations:



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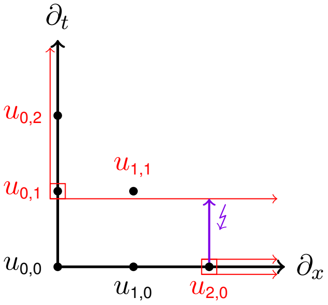
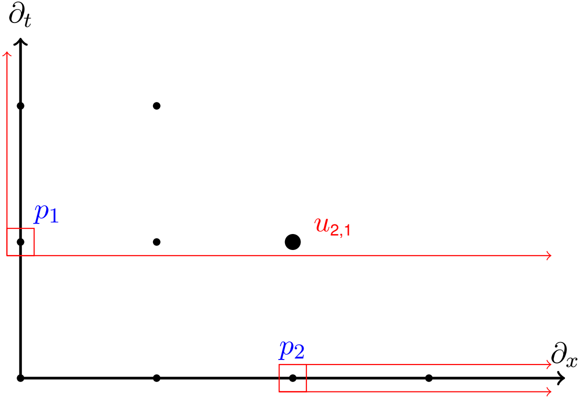


One has to consider $\partial_t u_{2,0} = u_{2,1}$ as new equation.

Differential Reduction

Reduce $u_{2,1}$ by $p_1 := u_{0,1} + u_{0,0}u_{1,0}$ and $p_2 := u_{2,0}$.



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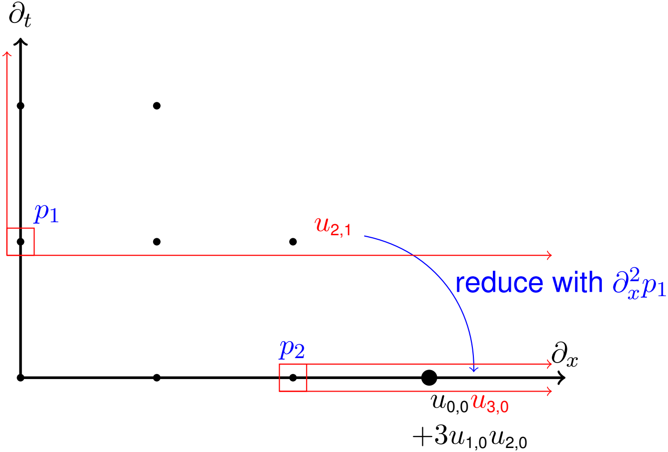

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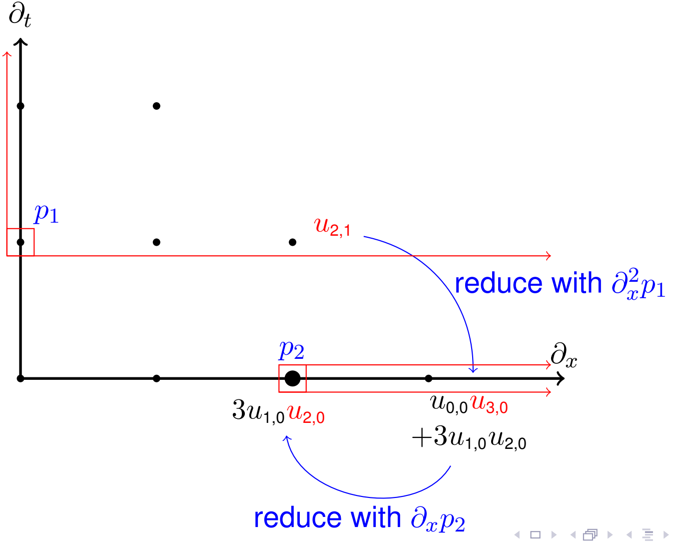

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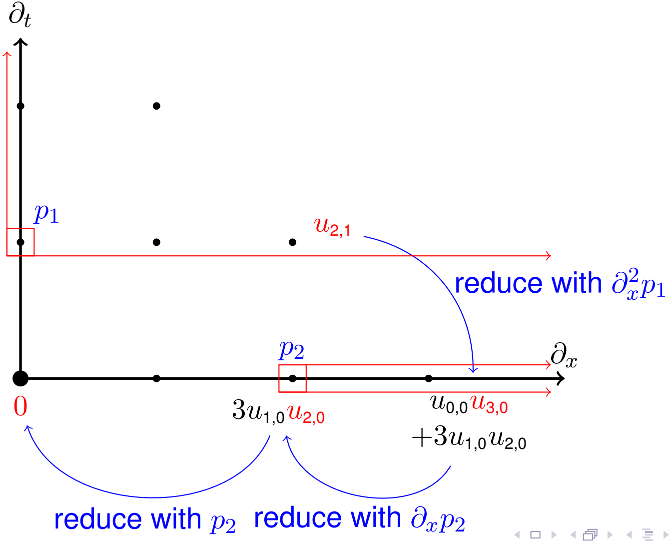

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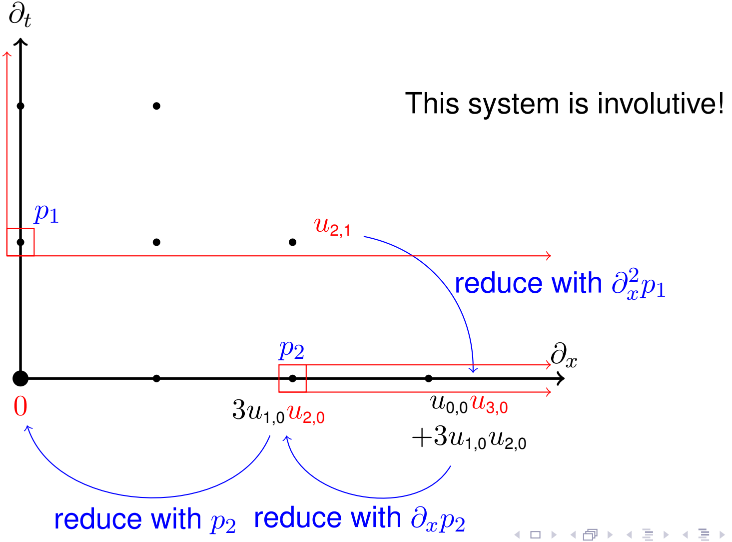

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- decide vanishing of equations on solutions by **reduction.**

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They

- contain all **compatibility conditions.**
- decide vanishing of equations on solutions by **reduction.**

The **differential Thomas decomposition** *disjointly* decomposes into differential simple systems.

Differential Simple Systems

Differential simple systems are defined as

- algebraically simple
- involutive
- minimal (in a technical sense)

They

- contain all **compatibility conditions.**
- decide vanishing of equations on solutions by **reduction.**

The **differential Thomas decomposition** *disjointly* decomposes into differential simple systems.

Suitable ranking allows **differential elimination.**

Example (Compatibility Condition)

The incompressible Navier-Stokes equations

$$\begin{aligned} v_t^{(1)} + \underline{v} \cdot \nabla v^{(1)} + p_x - \Delta v^{(1)} &= 0 \\ v_t^{(2)} + \underline{v} \cdot \nabla v^{(2)} + p_y - \Delta v^{(2)} &= 0 \\ v_t^{(3)} + \underline{v} \cdot \nabla v^{(3)} + p_z - \Delta v^{(3)} &= 0 \\ \nabla \cdot \underline{v} &= 0 \end{aligned}$$

yield the Poisson equation for the pressure

$$\Delta p = -\left(v_x^{(1)}\right)^2 - \left(v_y^{(2)}\right)^2 - \left(v_z^{(3)}\right)^2 - 2 v_y^{(1)} v_x^{(2)} - 2 v_z^{(1)} v_x^{(3)} - 2 v_z^{(2)} v_y^{(3)}$$

as compatibility condition.

Implementation

Available in the AlgebraicThomas and DifferentialThomas packages for MAPLE 11 and later:

`http://wwwb.math.rwth-aachen.de/thomasdecomposition/`

Timings - AlgebraicThomas

Example	RegularChains	RegSer	simser	AlgebraicThomas
Wang1	3.4	0.3	1.0	2.6
Wang2	7.2	7.2	8.6	6.3
Wang4	∞	∞	∞	0.3
Wang14	0.5	∞	∞	1.4
Wang16	0.8	1.4	1.7	1.9
Wang17	7.0	5.5	7.7	84.6
Wang19	459.4	0.4	0.5	0.5
Wang21	1.6	95.1	∞	5.7
Wang23	0.4	0.2	∞	29.3
Wang24	1.1	1.5	2.9	0.9

[1] All computations on third generation Opteron, 2.3 GHz, Linux x86_64, Maple 14, 4GB memory limit, times in seconds, stopped after 3600s

Timings - AlgebraicThomas

Example	RegularChains	RegSer	simser	AlgebraicThomas
Wang25	1.3	∞	∞	∞
Wang30	∞	∞	∞	41.0
Wang33	3.9	1.3	1.3	3.2
Wang35	1.6	1.4	1.5	1.9
Wang39	0.7	1.3	1.6	0.7
Wang41	1.5	1.8	2.1	7.5
Wang43	0.8	3.5	4.7	0.3
Wang44	24.2	4.0	5.4	1.4
Wang47	1.6	2.6	7.7	13.4
Wang49	0.4	634.5	36.2	0.6

[1] All computations on third generation Opteron, 2.3 GHz, Linux x86_64, Maple 14, 4GB memory limit, times in seconds, stopped after 3600s

Timings - AlgebraicThomas

Example	RegularChains	RegSer	simser	AlgebraicThomas
AlkashiSinus	0.6	0.1	7.0	5.3
Cheaters1	0.7	∞	∞	∞
Cheaters2	0.9	∞	∞	∞
Gerdt	1.7	2.6	3.0	10.0
Lazard	2.1	∞	∞	∞
Leykin-1	6.9	∞	∞	2.8
Maclane	2.9	∞	∞	9.5
Neural	0.5	∞	∞	1.2
Pavelle	1.6	∞	∞	∞
Wang93	1.4	2.1	4.5	6.3

[1] All computations on third generation Opteron, 2.3 GHz, Linux x86_64, Maple 14, 4GB memory limit, times in seconds, stopped after 3600s

Timings (ODE) - DifferentialThomas

Example	diffalg	DifferentialThomas
ODE 1	16.7[2]	1.9
ODE 2	∞	98.4
ODE 3	∞	70.3
diffalg 4	15.8	∞
diffalg 5	∞	0.5
Keppler 1	0.5	4.6
Keppler 2	0.7	14.6
Keppler 3	4.1	4.1
Murray 1	0.4	6.7
Murray 2	0.1	2.7

[2] All computations on third generation Opteron, 2.3 GHz, Linux x86_64, Maple 13, time in seconds, stopped after 600s

Timings (PDE) - DifferentialThomas

Example	diffalg	DifferentialThomas
PDE 1	4.2[3]	0.0
PDE 2	∞	0.1
PDE 3	∞	115.8
PDE 4	∞	25.7
Riquier 1	1.4	∞
Riquier 2a	6.4	3.7
Riquier 2b	0.8	8.4
Riquier 3	0.1	2.3
cyclic 6	∞	174.0
noon 6	∞	73.7
boulier	∞	8.8
diffalg 2	13.0	∞
diffalg 3	0.5	10.5

[3] All computations on third generation Opteron, 2.3 GHz, Linux x86_64, Maple 13, time in seconds, stopped after 600s

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