# Thomas Decomposition of Algebraic and Differential Systems

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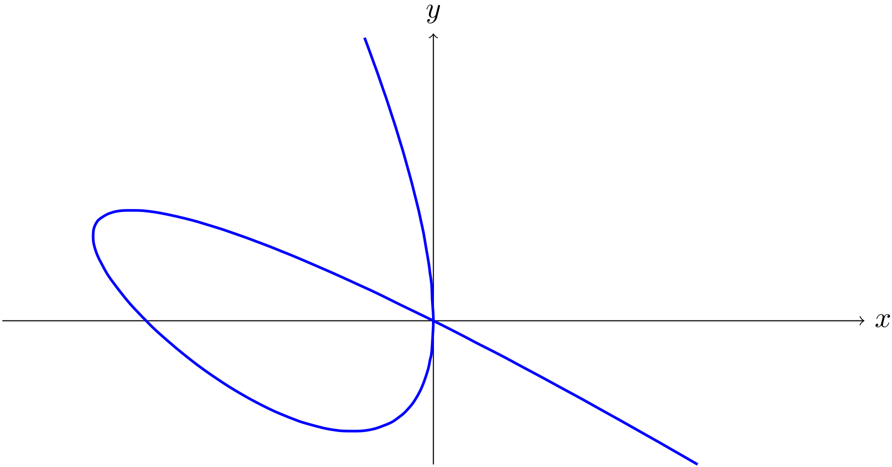
Joint work with Thomas Bächler, Vladimir Gerdt, Wilhelm Plesken, Daniel Robertz and Matthias Wisotzky.

# Outline

# Motivating Example

Decompose the variety

$$x^3 + 3x^2y + 3xy^2 + y^3 + x^2 + 2xy = 0$$



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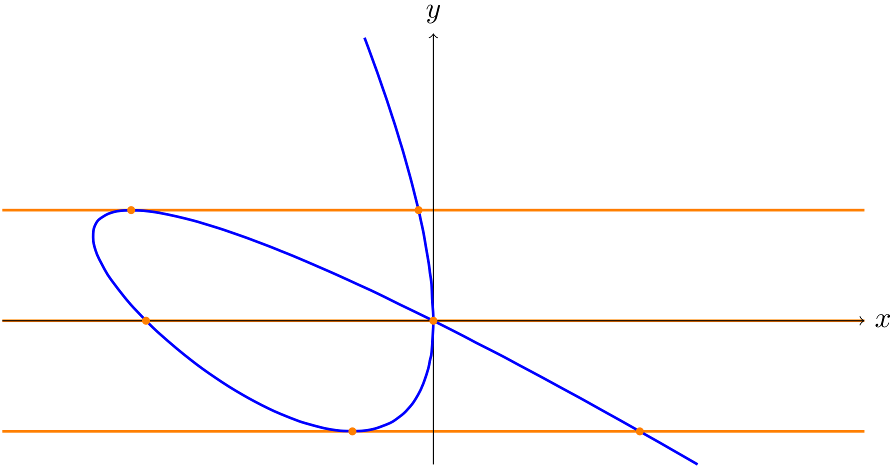

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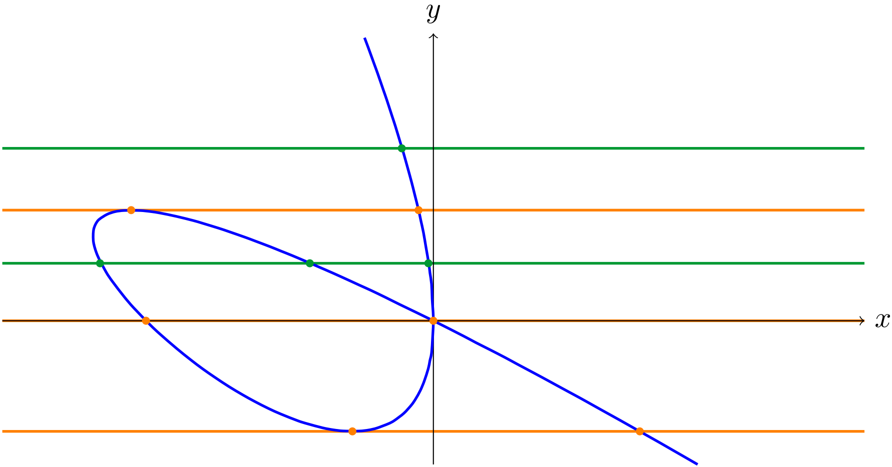

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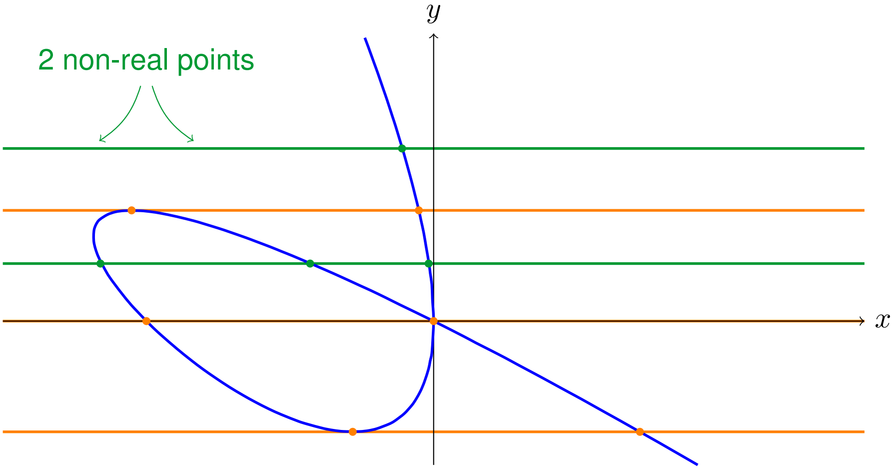
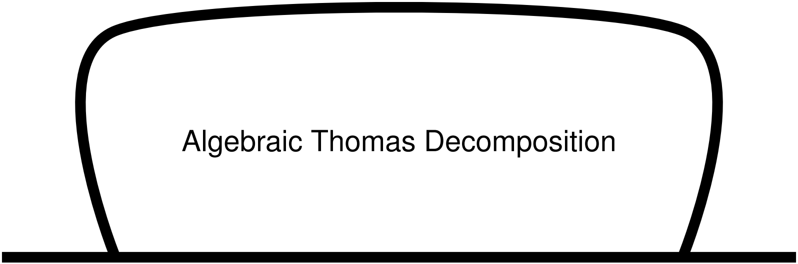


# History

- **1937:** Introduced by Joseph Miller Thomas in his book "Differential Systems".
- **1998:** Implementation by Dongming Wang for the algebraic case.
- **2007:** Gerdt: "On decomposition of algebraic PDE systems into simple subsystems".
- **2009:** Plesken: "Counting solutions of polynomial systems via iterated fibrations".
- **2009:** Implementation by Bächler and Lange-Hegermann for the algebraic and ordinary/partial differential case.


Algebraic Thomas Decomposition

# Definition

- Setup: System of polynomial equations and inequations over a field (characteristic $0$) with a fixed ordering $<$ on the variables.
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    - The **leading coefficient** does not vanish.
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- **Algebraic THOMAS decomposition**: *Disjoint* decomposition into simple systems.

# Simple Example

**Example:** $\{ax^2 + bx + c = 0\} \subseteq \mathbb{Q}[x, c, b, a]$.

THOMAS decomposition w.r.t. the ordering $a < b < c < x$:

$x$ $ax^2 + bx + c = 0$

$c$

$b$

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| $x$ | $ax^2 + bx + c = 0$ | $x$ | $2ax - b = 0$ | $x$ | $bx + c = 0$ |
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# Subresultants

During the algorithm we need to

- Replace several equations by their gcd.
- Divide equations by their gcd with an inequation.
- Make equations and inequations square-free.
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**Lemma (eg. Habicht, Mishra, . . . )**

The subresultants of two polynomials $p$, $q$ w.r.t. their highest variable provide conditions on the exact degree of their gcd. These conditions behave well under specializations.

Split the system using these conditions.

# Strict Treatment of Inequations

**Inequations** are treated as an integral part of the system.

$$\{x^2 - x + 1 = 0, x + a \neq 0\} \subseteq \mathbb{Q}[x, a], a < x$$

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$x$ : $x^2 - x + 1 = 0$

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$x$ : $x - (a + 1) = 0$

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Example: Reduce $x^2 + y$

$$\begin{array}{ll} {\color{red}x} \bullet & p_1 = y{\color{red}x}^2 - 1 = 0 \\ {\color{red}y} \bullet & p_2 = {\color{red}y}^2 + 1 = 0 \end{array}$$

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Decide vanishing of a polynomial on all solutions of the system.

# Example

$$p := x^3 + 3x^2y + 3xy^2 + y^3 + x^2 + 2xy$$

Thomas Decomposition of $\{p = 0\}$, $x > y$:

$$S_1 := \{p = x^3 + \ldots = 0, \quad \mathrm{disc}_x(p) = y^3 + \ldots \neq 0\}$$
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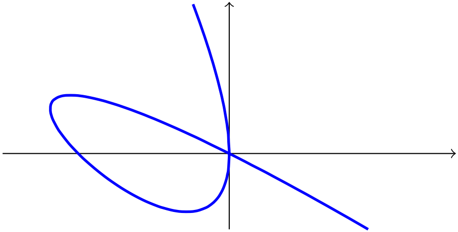

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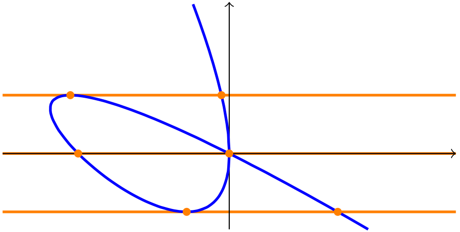
Fibers of the projection $\pi$ onto the $y$-component:

$$\left|\pi^{-1}(\{\hat{y}\})\right| = 2$$



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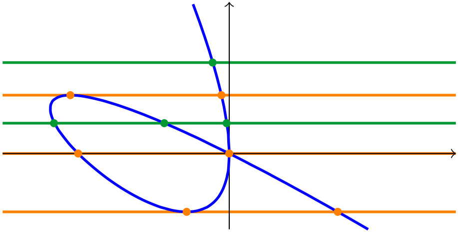
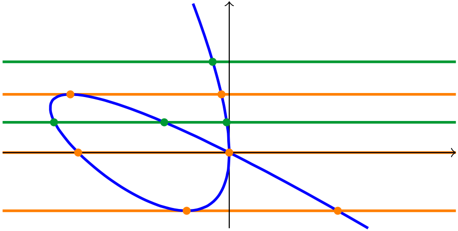


# The Counting Polynomial



$\underbrace{\text{There are three } x\text{-values}}_{3}$ $\underbrace{\text{for every } y\text{-value except three}}_{\infty-3}$.

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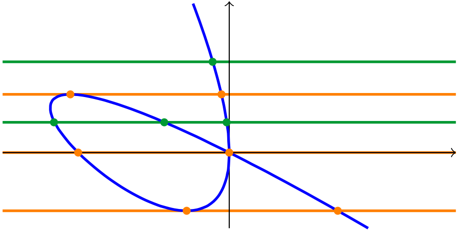

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This algebraic set has **counting polynomial**
$3 \cdot (\infty - 3) + 2 \cdot 3 = 3\infty - 3 \in \mathbb{Z}[\infty]$.

# The Counting Polynomial

$$\text{Multiply} \begin{cases} \text{the degree of the equation or} \\ \infty - (\text{the degree of the inequation}) \end{cases} \text{for every leader.}$$

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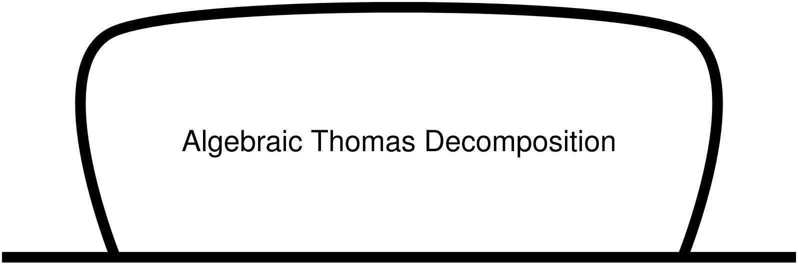
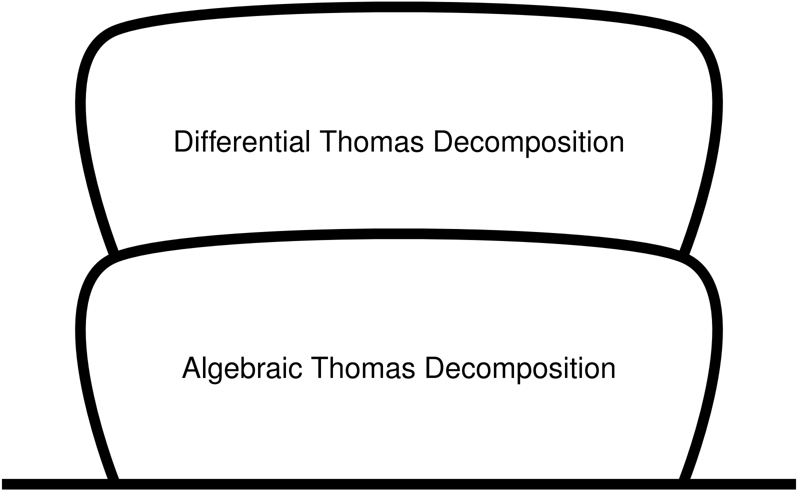
## Properties

- Does not depend on the decomposition
- Shows dimension and Euler characteristic
- Leads to finer invariants


Algebraic Thomas Decomposition


Differential Thomas Decomposition
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# Differential Algebra

- **Setting:**
  - Dependent variables $U = \{u^{(1)}, \ldots, u^{(m)}\}$ ("functions")
  - Independent variables $\{x_1, \ldots, x_n\}$ and derivations $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$
  - A differential field $F$ with $\mathrm{char}(F) = 0$

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- **Differential polynomial ring:**

$$F\{U\} = F[\ \underbrace{u_{\mathbf{i}}^{(j)}}_{\substack{\text{differential variables} \\ (\infty\text{-many})}}\ \mid j \in \{1 \ldots m\}, \mathbf{i} \in \mathbb{Z}_{\geq 0}^{n}\ ]$$

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- Differential equations $\rightsquigarrow F\{U\}$, e.g. BURGERS' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \rightsquigarrow u_{0,1} + u_{0,0}u_{1,0} \in F\{U\}$$

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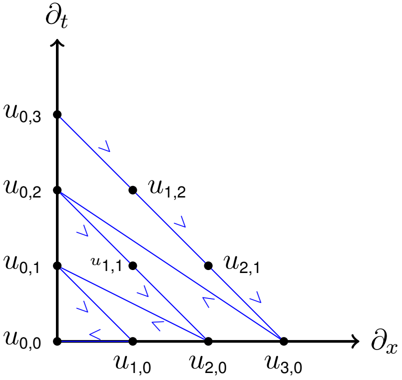
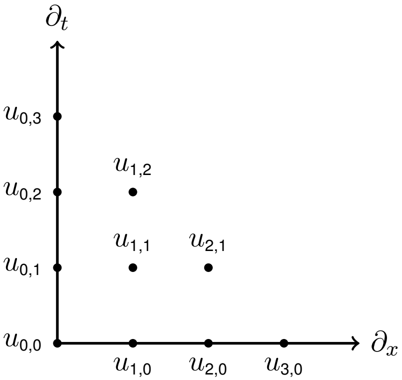
Example for a ranking: $u_{0,0} < u_{1,0} < u_{0,1} < u_{2,0} < u_{1,1} < u_{0,2} < u_{3,0} < \ldots$



# Differential Consequences

We indicate an equation in the picture by attaching it to its leader:

$$u_{0,1} + u_{0,0}u_{1,0}$$



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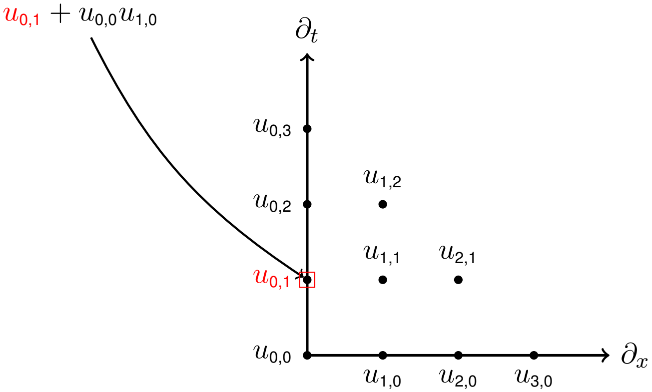
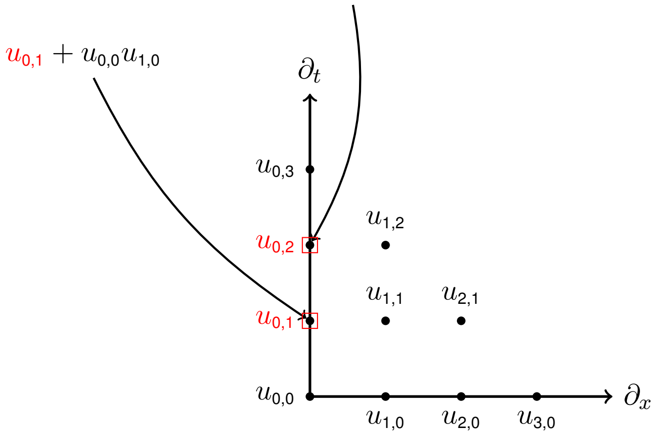


# Differential Consequences

A differential equation also affects the derivatives of the leader:

$$\partial_t(u_{0,1} + u_{0,0}u_{1,0}) = u_{0,2} + u_{0,1}u_{1,0} + u_{0,0}u_{1,1}$$



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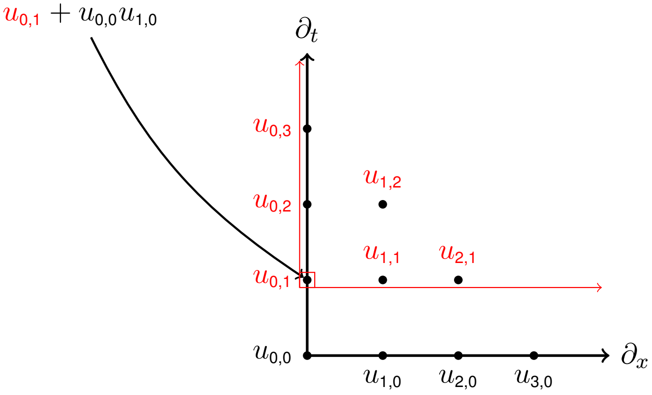

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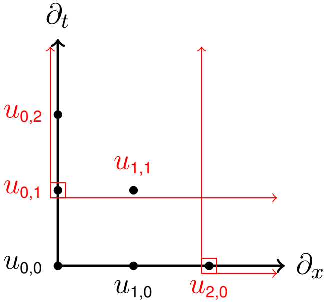

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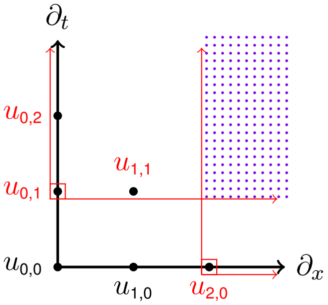
The second differential equation destroys triangularity:



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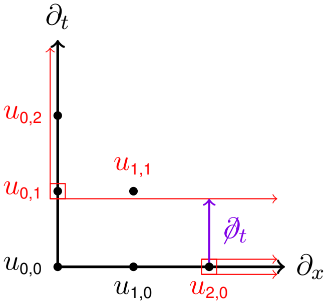
Use the involutive approach (Janet), which forbids certain derivations:



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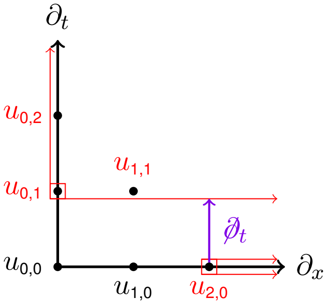
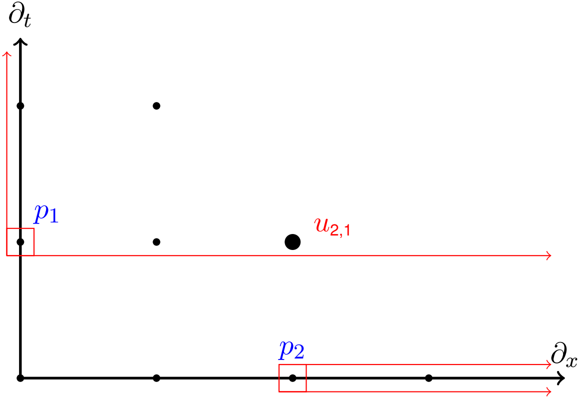



One has to consider $\partial_t u_{2,0} = u_{2,1}$ as new equation.

# Differential Reduction

Reduce $u_{2,1}$ by $p_1 := u_{0,1} + u_{0,0}u_{1,0}$ and $p_2 := u_{2,0}$.



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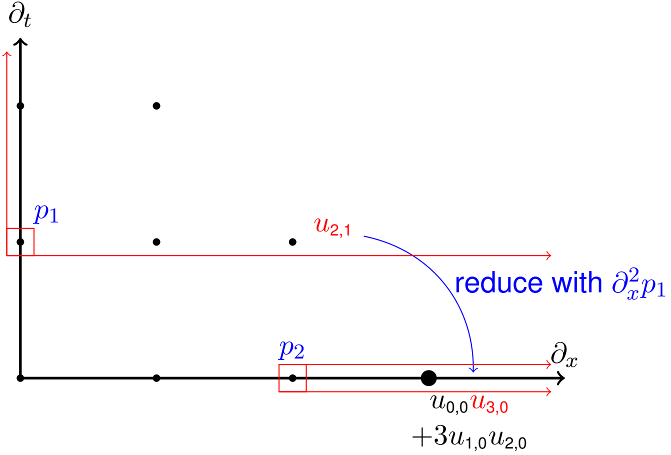

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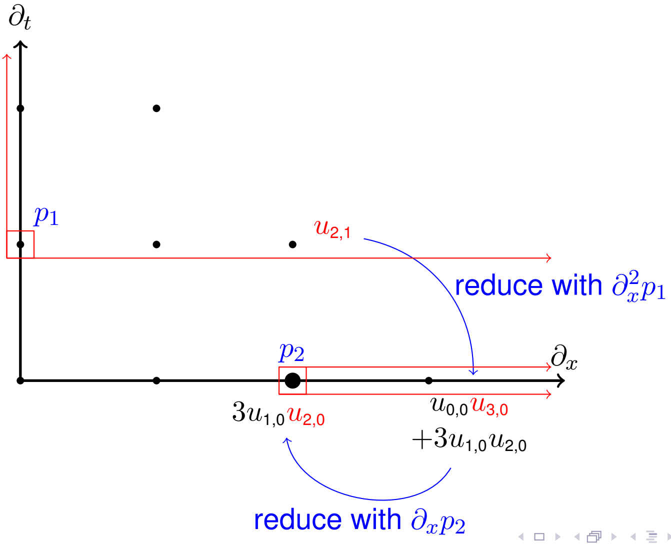

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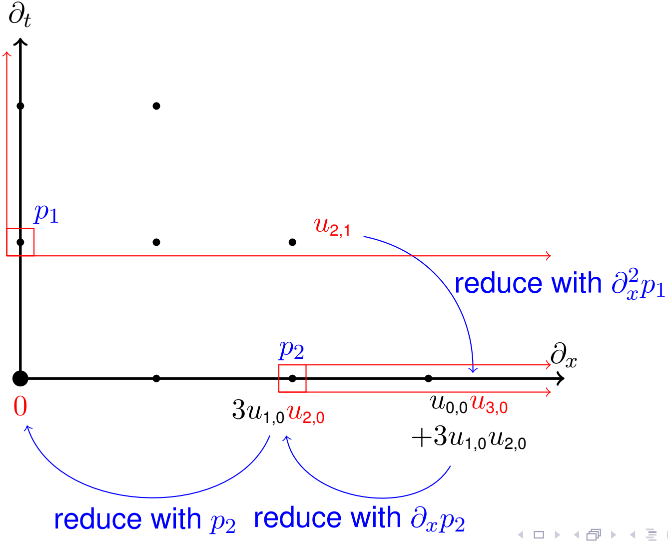

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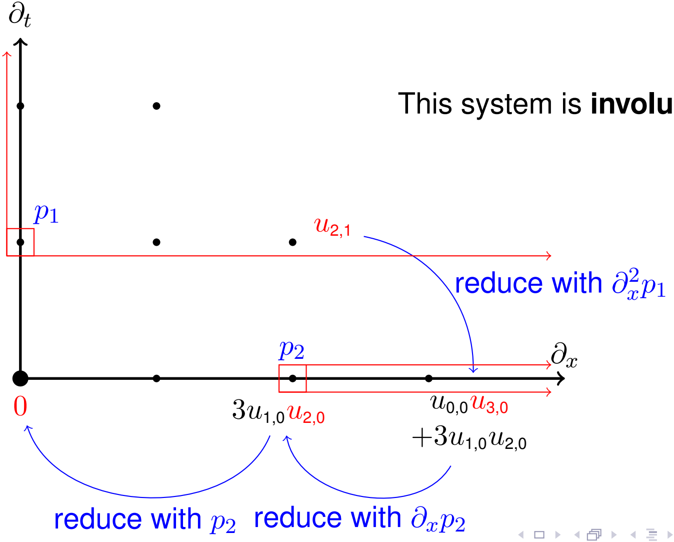


This system is **involutive**!

# Differential Simple Systems

**Differential simple systems** are defined as

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The **differential THOMAS decomposition** *disjointly* decomposes into differential simple systems.

Suitable ranking allows **differential elimination.**

# Example (Compatibility Condition)

The incompressible NAVIER-STOKES equations

$$\begin{aligned}
v_t^{(1)} + \underline{v} \cdot \nabla v^{(1)} + p_x - \Delta v^{(1)} &= 0 \\
v_t^{(2)} + \underline{v} \cdot \nabla v^{(2)} + p_y - \Delta v^{(2)} &= 0 \\
v_t^{(3)} + \underline{v} \cdot \nabla v^{(3)} + p_z - \Delta v^{(3)} &= 0 \\
\nabla \cdot \underline{v} &= 0
\end{aligned}$$

yield the POISSON equation for the pressure

$$\Delta p = -\left(v_x^{(1)}\right)^2 - \left(v_y^{(2)}\right)^2 - \left(v_z^{(3)}\right)^2 - 2v_y^{(1)}v_x^{(2)} - 2v_z^{(1)}v_x^{(3)} - 2v_z^{(2)}v_y^{(3)}$$

as compatibility condition.

# Example (Differential Elimination)

Predator–prey equations (LOTKA-VOLTERRA):

$$\begin{cases} \dot{x} = x\,(\alpha - \beta\, y) \\ \dot{y} = -y\,(\gamma - \delta\, x) \end{cases} \qquad \alpha, \beta, \gamma, \delta \ \text{ parameter}$$

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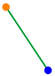
The generic case of the THOMAS decomposition:

$$\alpha = -\frac{x(y\ddot{y} - \dot{y}^2) - \dot{x}y\dot{y}}{xy\dot{y}}, \quad \beta = -\frac{y\ddot{y} - \dot{y}^2}{y^2\dot{y}}, \quad \gamma = \frac{x(y\ddot{y} - \dot{y}^2) - \dot{x}y\dot{y}}{\dot{x}y^2}, \quad \delta = \frac{y\ddot{y} - \dot{y}^2}{\dot{x}y^2}$$

# Example (Automatic Theorem Proving)

**Theorem:** Let $(X(t), Y(t))$ be the template curve with tractrix $(x(t), y(t))$ for distance $d > 0$ and non-degeneracy condition $\dot{x}\dot{Y} - \dot{X}\dot{y} \neq 0$.
The evolute of the tractrix is given by the intersection points of the normals of $(X(t), Y(t))$ and $(x(t), y(t))$.



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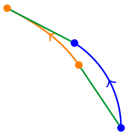

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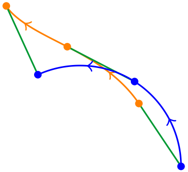

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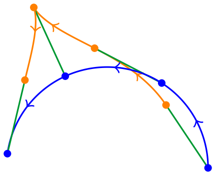

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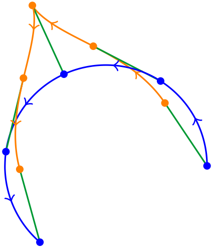

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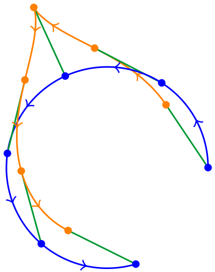

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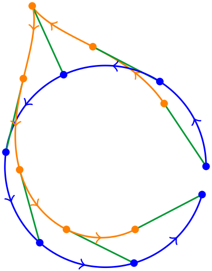

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Symbolically compute the intersection $a$ of the normals of template and tractrix and compute the evolute $b$ from the definition as envelope of the normals. The difference $a(t) - b(t)$ of these curves **reduces to zero** w.r.t. *all* systems of the above decomposition. □

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A refinement allows to count the number of power series solutions.

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**Effects of equations and inequations on TAYLOR coefficients:**

*All* power series coefficients have to satisfy an *equation*:

$u = 0 \rightsquigarrow$ every coefficient of a solution is zero. ☺

Only *one* coefficient has to satisfy an *inequation*:

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The power series coefficients of a solution are constrained *both* by the equation and the inequation. This violates triangularity.

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Take an inequation $q$ in $S$ and split $S$ into disjoint systems $S_1$ and $S_2$. In $S_1$ and $S_2$ different TAYLOR coefficients are affected by $q$.

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Finite number of coefficients not constrained by the equations
$\rightsquigarrow$ Triangularity after a finite number of splittings.

# Implementation

Available in the AlgebraicThomas and DifferentialThomas packages for MAPLE 11 and later:

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**Thanks for the attention and a happy $\pi$-day.**
**Remarks or Questions?**