LocalizeRingForHomalg

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Localization

- Let R be a commutative ring and $S \subseteq R$ a multiplicatively closed subset. Define the localization of R at S by $S^{-1}R := \{\frac{r}{s} \mid r \in R, s \in S\}/_\sim$
- Localize an R-module M by $S^{-1}R \otimes_R M$.
- Fact: Localization is an exact functor.

Localizing *"at"* a point:

Let $P \in \operatorname{Spec}(R)$, then $S := R \setminus P$ is mutiplicatively closed.
$R_P := S^{-1}R$ is a local ring.
Special case: $\mathfrak{m} \in \operatorname{MaxSpec}(R)$.

- Example: $R = K[x]$ and $\mathfrak{m} = \langle x \rangle$.
  Then $x \in \langle x - x^2 \rangle_{R_\mathfrak{m}}$, because $x = \frac{x-x^2}{1-x}$ or $(1-x)x = (x - x^2)$.

Syzygies

Assume that R

- ...is commutative
- ...is computable in `homalg`
- ...has $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ finitely generated

- Now let $A \in (R_{\mathfrak{m}})^{m \times n}$. Syzygies are given by a matrix $B \in (R_{\mathfrak{m}})^{k \times m}$ with $B \cdot A = 0$.

Corollary (`syz`):

We write $A = \frac{\tilde{A}}{\tilde{a}}$ with $\tilde{A} \in R^{m \times n}$ and $\tilde{a} \in R \setminus \mathfrak{m}$. For a syzygy matrix $\tilde{B} \in R^{k \times m}$ of $\tilde{A}$ the matrix $B := \frac{\tilde{B}}{1}$ is a syzygy matrix of A.
Proof: Localization is an exact functor. □

Module Membership

- Let $A \in R_{\mathfrak{m}}^{m \times n}$ and $B \in R_{\mathfrak{m}}^{1 \times n}$.
  Decide whether $B \in \langle A \rangle_{R_{\mathfrak{m}}} = R_{\mathfrak{m}}^{1 \times m} \cdot A$
- WLOG $A \in R^{m \times n}$ and $B \in R^{1 \times n}$

Theorem (DC0, dc0):

$B \in \langle A \rangle_{R_{\mathfrak{m}}} \Leftrightarrow B \in \langle A, \mathfrak{m}_1 B, \dots, \mathfrak{m}_k B \rangle_R$ for $\mathfrak{m} = \langle \mathfrak{m}_1, \dots, \mathfrak{m}_k \rangle$.

- Example: $R_{\mathfrak{m}} = K[x]_{\langle x \rangle}$.
  Compute with $B = x$ and $A = x - x^2$:
  $\langle x - x^2, x \cdot x \rangle_{K[x]} = \langle x \rangle_{K[x]} \ni x$
  $\Rightarrow x = (x - x^2) + x \cdot x$
  $\Leftrightarrow x - x \cdot x = (x - x^2)$
  $\Leftrightarrow (1 - x)x = (x - x^2)$

Easy Corollary for Complexity Theory

- Fact: The ideal membership problem of $\mathbb{Q}[x_1, \ldots, x_n]$ is EXPSPACE complete.

Corollary

The ideal membership problem of $\mathbb{Q}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$ is in EXPSPACE.

Implementation

Technical advantages of this project

- Local computations take place in the global ring.
- No local basis computation.

- Implemented as a GAP-package `LocalizeRingForHomalg`. On the one hand it uses `homalg` and on the other hand it provides representations for categories from `homalg`:

  `homalg` $\Longleftrightarrow$ `LocalizeRingForHomalg`

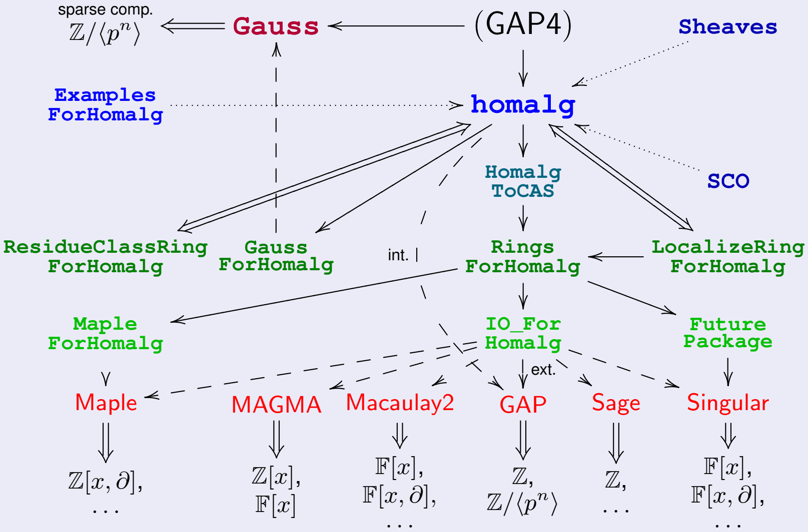
Matrices with denominators.

- Also implements an interface to Mora's algorithm for local polynomial rings in Singular:

  `RingsForHomalg` $\longleftarrow$ `LocalizeRingForHomalg`

"Our" Singular outperfoms Singular with MORA.

Current packages of the homalg project



For details see:

Mohamed Barakat, Markus Lange-Hegermann, *"An Axiomatic Setup for Algorithmic Homological Algebra and an Alternative Approach to Localization"* (`http://arxiv.org/abs/1003.1943`)